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On the existence and global structure of solutions for a class of fractional feedback control systems

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Abstract

In this paper, using topological tools, guiding functions and bifurcation theory, we deal with the existence of a connected subset of nontrivial solutions of a system whose dynamics of the system and feedback law are expressed in the form of fractional differential equations.

Keywords: Bifurcation theory, boundary value problem, Caputo derivative, degree theory, Fredholm operators, guiding functions.
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1. Introduction

Global structure of solutions for a boundary value problem (BVP in short), which arises naturally from some physical and control problems, attracts the attention of many researchers. Various methods have been investigated by researchers for the existence of solutions to the BVPs. One of them is evaluating the topological degree of an integral operator whose fixed point set coincides with the set of solutions. M. A. Krasnoselskii et.al. [8], evaluated this quantity through the evaluating the topological index of the especial "guiding function" of the equation under consideration. The method of guiding functions was also applied to study the global bifurcation problem for differential inclusions in various research papers (see e.g. [9, 10, 14, 13]).

In this paper by applying the method of guiding functions we obtain the global structure of the solution set of the following feedback control system, whose first equation describes the dynamics of the system and the second inclusion represents the feedback,

$$\begin{cases} D^{\beta}_{0^+}x(t) - a\mu x(t) = f(t, x(t), u(t), \mu) \qquad t \in I, \\ D^{\alpha}_{0^+}u(t) \in G(t, x(t), u(t), \mu), \\ x(0) = x(T), \quad u(0) = 0. \end{cases} \tag{1.1}$$

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where $I = [0, T]$, $f : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a continuous map, $G : I \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to Kv(\mathbb{R}^m)$ is a multivalued map, $Kv(\mathbb{R}^m)$ denotes the collection of all nonempty, compact convex subsets of $\mathbb{R}^m$, $D^{\alpha}_{0+}u(t)$ and D^{β}_{0+} are standard Caputo derivatives for $n-1 < \alpha \leq n$ and $\frac{1}{2} < \beta \leq 1$, $x : I \to \mathbb{R}^n$ is a trajectory of the system and $u : I \to \mathbb{R}^m$, is a control function.
The existence of solutions for a class of feedback control systems of second order differential equation in Hilbert spaces is studied by Loi et.al. [12] and also the global bifurcation problem for such systems is investigated by Loi [11], but there is no paper studying such problems with fractional-order (FO) controller, so the purpose of present paper is to fill this gap.
To the best of our knowledge, this is probably the first effort to investigate the existence of nontrivial solutions for BVPs in a system with (FO) dynamics and feedback law. Since (FO) controller providing more flexibility than integer orders in the design, applying them will be more better to describe physical and control problems.

The paper is organized as follows. The next section contains background materials and preliminaries from multivalued analysis. In section 3, we present some necessary notations definitions and lemmas from fractional calculus and fractional Sobolev spaces. Section 4 contains several properties of Fredholm operators. In section 5, some results on bifurcation theory and topological index are listed. In the last section, the main results of present paper are given and proved.

2. Preliminaries

Let X, Y be metric spaces. Denote by $P(Y)[C(Y), K(Y)]$ the collection of all nonempty [resp., nonempty closed, nonempty compact] subsets of Y . For a Banach space E by symbols $Cv(E)[Kv(E)]$ we denote the collection of all nonempty convex closed [resp., nonempty convex compact] subsets of E.

Definition 2.1 ([2])**.** A multivalued map (multimap) $F : X \to P(Y)$ is said to be compact if $F(X)$ is relatively compact in Y and is said to be upper semicontinuous (u.s.c.) if for every open subset $V \subset Y$ the set

$$F_{+}^{-1}(V) = \{x \in X : F(x) \subset V\}$$

is open in X. A u.s.c. multimap F is said to be completely u.s.c., if it maps every bounded subset $X_1 \subset X$ into a relatively compact subset $F(X_1)$ of Y.

Definition 2.2. A set $M \in K(Y)$ is said to be aspheric (see, e.g., [14, 24]), if for every $\varepsilon > 0$ there exists $\delta > 0$ such that each continuous map $\sigma : S^n \to O_\delta(M)$, $n = 0, 1, 2, ...,$ can be extended to a continuous map $\tilde{\sigma} : B^{n+1} \to O_\varepsilon(M)$, where $S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$, $B^{n+1} = \{x \in \mathbb{R}^{n+1} : ||x|| \leq 1\}$, and $O_\delta(M)[O_\varepsilon(M)]$ denotes the δ-neighborhood [resp. ε-neighborhood] of the set M.

Definition 2.3 ([5])**.** A nonempty compact space A is said to be an R_δ-set if it can be represented as the intersection of a decreasing sequence of compact, contractible spaces.

Definition 2.4 ([2])**.** A u.s.c. multimap $\Sigma : X \to K(Y)$ is said to be a J-multimap ($\Sigma \in J(X, Y)$) if every value $\Sigma(x)$, $x \in X$, is an aspheric set.

Now let us recall (see, e.g., [2]) that a metric space X is called the absolute retract (the AR-space) [resp., the absolute neighborhood retract (the ANR-space)] provided for each homeomorphism h taking it onto a closed subset of a metric space $\acute{X}$, the set $h(X)$ be the retract of $\acute{X}$ [resp., of its open neighborhood in $\acute{X}$]. Notice that the class of ANR-spaces is broad enough: in particular, a finite-dimensional compact set is the ANR-space if and only if it is locally contractible. In turn, it means that compact polyhedrons and compact finite-dimensional manifolds are the ANR-spaces. The union of a finite number of convex closed subsets in a normed space is also the ANR-space.

Proposition 2.5 ([2])**.** *Let Z be an ANR-space. In each of the following cases a u.s.c. multimap $\Sigma : X \to K(Z)$ is a J -multimap: for each $x \in X$ the value $\Sigma(x)$ is*

(*a*) *a convex set;*
(*b*) *a contractible set;*
(*c*) *an* R_δ *-set;*
(*d*) *an* AR*-space.*

In particular, every continuous map $\sigma : X \to Z$ is a J-multimap.

Definition 2.6. Let $O \subset X$. By $J^c(O, X)$ we will denote the collection of all multimaps $F : O \to K(X)$ that may be represented in the form of composition $F = fog$, where $g \in J(O, Y)$ and $f \in J(Y, X)$. The composition fog will be called the decomposition of F. We will denote $F = (fog)$.

It has to be noted that a multimap can admit different decompositions (see [2]).
Now, let X be a Banach space and $U \subset X$ be an open bounded subset and $F = (foG) \in J^c(\overline{U}, X)$ be a completely continuous multimap such that $x \notin F(x)$ for $x \in \partial U$. Then the topological degree $deg(i - F, U)$ of the corresponding multivalued vector field $(i-F)(x) = x-F(x)$ is well defined and has all usual properties of the Leray Shauder topological degree (see, e.g., [2]).

3. Fractional Calculus

Now for convenience, we present some definitions and results from fractional calculus which can be found in [7].

Definition 3.1. The fractional integral of arbitrary order $0 < \alpha \in \mathbb{R}$ of a function $h : (0, 1) \to \mathbb{R}$ is defined by

$$I^{\alpha}_{0+}h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds,$$

provided that the right hand side integral is pointwise defined on $(0, +\infty)$.

Definition 3.2. For a continuous function $h : (0, 1) \to \mathbb{R}$, the αth ($\alpha > 0$) Caputo fractional derivative of h, is given by

$$D^{\alpha}_{0+}h(t) = I^{n-\alpha}_{0+}\frac{d^n h(t)}{dt^n} = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds.$$

Here n is the smallest integer which satisfies $n - 1 < \alpha \leq n$.

Lemma 3.3. Let $n - 1 < \alpha \leq n$. The fractional differential equation $D^{\alpha}_{0+}h(t) = 0$ has solution

$$h(t) = C_1 + C_2 t + C_3 t^2 + \cdots + C_n t^{n-1}.$$

Lemma 3.4. Assume that $h(t)$ with a fractional derivative of order $\alpha > 0$. Then

$$I^{\alpha}_{0+}D^{\alpha}_{0+}h(t) = h(t) + C_1 + C_2 t + C_3 t^2 + \cdots + C_n t^{n-1}, \quad C_i \in R, i = 1, 2, \ldots, n,$$

where n is the smallest integer which satisfies $n - 1 < \alpha \leq n$.

3.1. Fractional Sobolev Spaces

Fractional Sobolev spaces form a very important class of Banach spaces which proved to be a powerful tool in the analysis of fractional boundary value problems. In order to define fractional Sobolev spaces, first one need to generalize the classical notion of Sobolev space. As usual, we denote by $C(I, \mathbb{R}^n)(L_p(I, \mathbb{R}^n))$ the space of all continuous (resp. the space of all $p-$summable) functions $x : I \to \mathbb{R}^n$. For every $u \in C(I, \mathbb{R}^n)$ and $f \in L_p$ their corresponding norms are:

$$||u||_C = \max_{t\in I}|u(t)| \quad \text{and} \quad ||f||_p = (\int_0^T |f(s)|^p ds)^{\frac{1}{p}}.$$

The symbol $\langle .,. \rangle$ denotes the inner product in $L_2(I,\mathbb{R}^n)$ and $B_C(0,R)$ denotes the ball in $C(I,\mathbb{R}^n)$ of radius R centered at the origin. For an integer $k>0$ the Sobolev space $W^{k,p}$ is defined as

$$W^{k,p}(I,\mathbb{R}^n)=\{u\in L_p(I,\mathbb{R}^n): u^{(m)}\in L_p(I,\mathbb{R}^n) \ \text{ for all } \ 0\le m\le k\}, \tag{3.1}$$

where $u^{(m)}$ denotes the distributional derivative of u of order m. In other words, $W^{k,p}(I,\mathbb{R}^n)$ consists of all the $L_P(I,\mathbb{R}^n)$ functions such that all distributional derivatives up to order k belong to $L_P(I,\mathbb{R}^n)$. This space is equipped by the norm

$$||u||_W=(\sum_{m=0}^{k}||u^{(m)}||_p^p)^{\frac{1}{p}}.$$

In particular the norm of an element $u\in W^{1,2}(I,\mathbb{R}^n)$ is defined as

$$||u||_W=(||u||_2^2+||\acute{u}||_2^2)^{\frac{1}{2}}.$$

By $W_T^{k,p}(I,\mathbb{R}^n)$ we will denote the subset of $W^{k,p}(I,\mathbb{R}^n)$ consisting of all functions u such that $u(0)=u(T)$ i.e.,

$$W_T^{k,p}(I,\mathbb{R}^n)=\{u\in W^{k,p}(I,\mathbb{R}^n): u(0)=u(T)\}.$$

According to the Sobolev embedding theorem (see e.g. Theorems 3.9.50 and 3.9.53 [1]) the space $W^{1,2}(I,\mathbb{R}^n)$ is compactly embedded into $C(I,\mathbb{R}^n)$.
The fractional extension of (3.1) for any $s\in\mathbb{R}$ will be the spaces $W^{s,p}(I,\mathbb{R}^n)$ (see Definition 3.22 in [3]),

$$W^{s,p}(I,\mathbb{R}^n)=\{u\in L_p(I,\mathbb{R}^n): D^{\alpha}u\in L_p(I,\mathbb{R}^n) \ \text{ for all } \ 0\le\alpha\le s\}, \tag{3.2}$$

where $D^{\alpha}u$ denotes the Caputo fractional derivative of u or equivalently

$$W^{s,p}(I,\mathbb{R}^n)=\{u\in L_p(I,\mathbb{R}^n): \int_I\int_I\frac{|u(x)-u(y)|^p}{|x-y|^{1+sp}}dxdy<\infty\}.$$

This extension especially for $0<s\le 1$ is of great service in connection with boundary value problems. Now according to [1] and Theorem 4.17 in [3], the embedding $W^{s,2}(I,\mathbb{R}^n)$ into $C(I,\mathbb{R}^n)$ is compact provided $s>\frac{1}{2}$.

4. Fredholm Operators

Now, we recall some basic notions of the theory of linear Fredholm operators. Let X and Y be Banach spaces.

Definition 4.1. A bounded linear operator $L: X\to Y$ is said to be a Fredholm operator of index zero, if
$(1i)$ ImL is closed in Y;
$(2i)$ $KerL$ and $CokerL$ have finite dimensions and $dimKerL=dimCokerL$.

Let $L: domL\subset X\to Y$ be a linear Fredholm operator of index zero. Then there exist projections $P: X\to X$ and $Q: Y\to Y$ such that $ImP=KerL$ and $KerQ=ImL$. If the operator

$$L_P: domL\cap KerP\to ImL$$

is defined as the restriction of L on $domL\cap KerP$, then L_P is a linear isomorphism and so the linear operator $k_P: ImL\to domL$, $k_P=L_P^{-1}$ is well-defined. Now, let $CokerL=Y/ImL$. Define a canonical projection operator $\pi_L: Y\to CokerL$,

$$\pi_L(z)=z+ImL,$$

and let $l_L: CokerL\to KerL$ be a linear continuous isomorphism. Then, the equation

$$Lx=y, \quad y\in Y,$$

is equivalent to the following relation:

$$x = Px + (l_L \pi_L + k_L)y, \tag{4.1}$$

where $k_L : Y \to X$ is defined as

$$k_L = k_P(i - Q).$$

Now Define

$$A : W_T^{\beta,2}(I, \mathbb{R}^n) \to L_2(I, \mathbb{R}^n)$$

$$Ax = D^\beta x. \tag{4.2}$$

Lemma 4.2. *Let A be defined by (4.2), then*

$$KerA = \{x \in W_T^{\beta,2}(I, \mathbb{R}^n) | x(t) = c \in \mathbb{R}^n\}$$

$$ImA = \{y \in L_2(I, \mathbb{R}^n) | \int_0^T \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} y(s)ds = 0\}.$$

Proof. Let $D^\beta x = 0$, then from Lemma 3.3 $KerA = \mathbb{R}^n$. If $y \in ImA$, then there exists a function $x \in domA$ such that $y(t) = D^\beta x(t)$. So $x(t) = I_{0+}^\beta y(t) + c_0$, therefore from $x(0) = x(T)$ one can concludes that

$$\int_0^T \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} y(s)ds = 0. \tag{4.3}$$

On the other hand, suppose $y \in Y$ and satisfies (4.3). Let $x(t) = I_{0+}^\beta y(t)$; then $x \in domA$ and $Ax(t) = D_{0+}^\beta x(t) = y(t)$. So that, $y \in ImA$. The proof is complete. □

Lemma 4.3. *Let A be defined by (4.2), $X = W_T^{\beta,2}(I, \mathbb{R}^n)$ and $Y = L_2(I, \mathbb{R}^n)$ then A is a Fredholm operator of index zero, and the linear continuous projector operators $P : X \to X$ and $Q : Y \to Y$ can be defined as*

$$Px(t) = x(0), \qquad Qy(t) = \frac{\beta}{T^\beta} \int_0^T (T-s)^{\beta-1} y(s)ds, \quad \textit{for all} \quad t \in I.$$

Proof. For any $y \in Y$, we have

$$Q^2 y = Qy \frac{\beta}{T^\beta} \int_0^T (T-s)^{\beta-1} ds = Qy.$$

Let $y_1 = y - Qy$, then we get that

$$\frac{\beta}{T^\beta} \int_0^T (T-s)^{\beta-1} y_1(s)ds = \frac{\beta}{T^\beta} \int_0^T (T-s)^{\beta-1} y(s)ds - \frac{\beta}{T^\beta} \int_0^T (T-s)^{\beta-1} Qy(s)ds = Qy(t) - Q^2 y(t) = 0,$$

which implies $y_1 \in ImA$. Hence $Y = ImA + ImQ$. Since $ImA \cap ImQ = \{0\}$, we have $Y = ImA \oplus ImQ$. Thus $n = dimKerA = dimImQ = codimImA$. This means that A is a Fredholm operator of index zero. □

Corollary 4.4. *Let A be defined by (4.2), then*

$$KerA \cong \mathbb{R}^n \cong CokerA,$$

and A is a Fredholm operator of index zero.

5. Bifurcation theorem

Let X be a Banach space. Denote by $B_X(0,r)$ the ball of radius r centered at 0 in X. Consider the following one-parameter family of inclusions:

$$x \in F(x,\mu), \tag{5.1}$$

where $F : X \times R \to K(X)$ is a completely u.s.c. J^c-multimap satisfying the following conditions:
$(F1)$ $0 \in F(0,\mu)$ for all $\mu \in \mathbb{R}$;
$(F2)$ for each $\mu, 0 < |\mu - \mu_0| \leq r_0$, there is $\delta_\mu > 0$ such that $x \notin F(x,\mu)$ when $0 < ||x|| \leq \delta_\mu$, where μ_0, r_0 are given numbers.
A point $(0,\mu)$ is said to be a bifurcation point of inclusion (5.1), if for every open subset $U \subset X \times \mathbb{R}$ with $(0,\mu_*) \in U$, there exists a point $(x,\mu) \in U$ such that $x \neq 0$ and $x \in F(x,\mu)$.
From $(F2)$, it follows that for each $\mu, 0 < |\mu - \mu_0| \leq r_0$ the topological degree

$$deg(i - F(.,\mu), B_X(0,\delta_\mu)),$$

is well defined. Then, the bifurcation index of the multimap F at $(0,\mu_0)$ may be defined as

$$\begin{aligned} Bi[F,(0,\mu_0)] = & \lim_{\mu\to\mu_0^+} deg(i - F(.,\mu), 0, B_X(0,\delta_\mu)) \\ - & \lim_{\mu\to\mu_0^-} deg(i - F(.,\mu), 0, B_X(0,\delta_\mu)). \end{aligned} \tag{5.2}$$

Let us denote by S the set of all non-trivial solutions to inclusion (5.1), i.e., $S = \{(x,\mu) \in X \times \mathbb{R}; x \neq 0 \quad \text{and} \quad x \in F(x,\mu)\}$. The following assertion can be easily followed from the global bifurcation theorems presented in [9].

Theorem 5.1. *Under conditions $(F1),(F2)$, assume that $Bi[F,(0,\mu_0)] \neq 0$. Then, there exists a connected subset $C \subset S$ such that $(0,\mu_0) \in \overline{C}$ and one of the following occurs:*
(a) C is unbounded;
(b) $(0,\mu_*) \in \overline{C}$ for some $\mu_* \neq \mu_0$.

6. Main Results

Consider the BVP (1.1) with the following assumptions:
$(f1)$ there exists $0 < c < a$ such that

$$|f(t,x,y,z)| \leq c|x|(|y| + |z|),$$

for all (x,y,z) and $t \in I$,
$(g1)$ for a.e. $t \in I$, the multimap $G(t,.,.,.) : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to Kv(\mathbb{R}^m)$ is u.s.c.,
$(g2)$ the multimap G is uniformly continuous with respect to the second and fourth arguments in the following sense: For every $\varepsilon > 0$, there is $\delta > 0$ such that

$$G(t,x,y,z) \subset O_\varepsilon(G(t,\overline{x},y,\overline{z})), \forall (t,y) \in I \times \mathbb{R}^m,$$

provided $\max\{|\overline{x} - x|, |\overline{z} - z|\} < \delta$;
$(g3)$ there is $B > 0$ such that $Be^{TB} < \frac{(a-c)\Gamma(\alpha)}{cT^\alpha}$ and

$$|G(t,x,y,z)| \leq \beta(|x| + |y| + |z|),$$

for all $(x,y,z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ and a.e. $t \in T$,
$(g4)$ for every $(x,y,z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ the multifunction $G(.,x,y,z) : I \to Kv(\mathbb{R}^m)$ has a measurable

selection.
By Theorem 1.3.5 [6] for each $w \in \mathbb{R}^m$ and $(x, \mu) \in C(I, \mathbb{R}^n) \times \mathbb{R}$, the multifunction $G^{(x,\mu)}(., w)$ defined as

$$G^{(x,\mu)} : I \times \mathbb{R}^m \to Kv(\mathbb{R}^m),$$

$$G^{(x,\mu)}(t, w) = G(t, x(t), w, \mu),$$

has a measurable selection. Moreover, from $(g1)$ and $(g2)$, for a.e. $t \in I$ the multimap $G^{(x,\mu)}(t, w)$ depends upper semicontinuously on (x, w, μ).
Now, we claim that for each $(x, \mu) \in C(I, \mathbb{R}^n) \times \mathbb{R}$, the set $\Pi^{(x,\mu)}$ of solutions of the inclusion problem

$$\begin{cases} D^{\alpha}_{0+}u(t) \in G(t, x(t), u(t), \mu) \qquad t \in (0, 1), \\ \\ u(0) = 0. \end{cases} \tag{6.1}$$

is an R_δ-set in $C(I, \mathbb{R}^m)$ and the multimap

$$\Pi : C(I, \mathbb{R}^n) \times \mathbb{R} \to K(C(I, \mathbb{R}^m)),$$

$$\Pi(x, \mu) = \Pi^{(x,\mu)}$$

is upper semi-continuous.
By a solution of (6.1) we shall understand an absolutely continuous map $u : I \to \mathbb{R}^m$ such that $D^{\alpha}u(t) = g(t, u(t))$ for almost all $t \in I$, $u(0) = 0$ and $g \in G^{(x,\mu)}$.
Define the fractional integral operator $F : C(I, \mathbb{R}^m) \to C(I, \mathbb{R}^m)$ by

$$F(u)(t) = I^{\alpha}_{0+}g(r, u(r))dr,$$

for every $u \in C(I, \mathbb{R}^m)$ and $t \in I$. Then $Fu(0) = 0$ and $D^{\alpha}_{0+}F = g(t, u(t))$, for a given map $g \in G^{(x,\mu)}$. It is easy to see that F satisfies all the assumptions of Theorem (69.2) in [2], so by applying Theorems (69.2) and (69.9) in [2] and using the same argument in section 69 of [2], the desired result is obtained.
By a solution to problem (1.1), we mean a pair $(x, \mu) \in W^{\beta,2}(I, \mathbb{R}^n) \times \mathbb{R}$ such that there is $u \in \Pi(x, \mu)$ that the following holds

$$D^{\beta}_{0+}x - a\mu x(t) = f(t, x(t), u(t), \mu) \quad \text{ for a.e.} \quad t \in I.$$

Note that by $(f1)$, $(0, \mu)$ is a solution of (1.1) for every $\mu \in \mathbb{R}$. These solutions are called trivial. Here, the set of all non-trivial solutions of (1.1) is denoted by S . In what follows, we need the following statement.

Lemma 6.1 ([4])**.** *Let $u, v : [a, b] \to \mathbb{R}$ be continuous nonnegative functions and $C \geq 0$ be a constant and*

$$v(t) \leq C + \int_a^t u(s)v(s)ds, \quad a \leq t \leq b.$$

Then

$$v(t) \leq Ce^{\int_a^t u(s)ds}, \qquad a \leq t \leq b.$$

Theorem 6.2. *Let conditions $(f1)$ and $(g1) - (g4)$ hold. Then, there is an unbounded connected subset $C \subset S$ such that $(0, 0) \in \overline{C}$.*

Proof. For every $(x, \mu) \in C(I, \mathbb{R}^n) \times \mathbb{R}$, define the following multimap:

$$\overline{\Pi} : C(I, \mathbb{R}^n) \times \mathbb{R} \to K(C(I, \mathbb{R}^n)) \times C(I, \mathbb{R}^m) \times \mathbb{R}),$$

$$\overline{\Pi}(x, \mu) = \{x\} \times \Pi(x, \mu) \times \{\mu\},$$

and a map $\overline{f} : C(I, \mathbb{R}^n) \times C(I, \mathbb{R}^m) \times \mathbb{R} \to L_2(I, \mathbb{R}^n)$

$$\overline{f}(x, u, \mu)(t) = a\mu x(t) + f(t, x(t), u(t), \mu), \quad t \in I.$$

Set $B : C(I, \mathbb{R}^n) \times \mathbb{R} \to K(L_2(I, \mathbb{R}^n))$,

$$B(x, \mu) = \overline{f} \circ \overline{\Pi}(x, \mu).$$

Since $\overline{\Pi}$ is a J-multimap and $\overline{f}$ is a continuous map, B is a J^c-multimap and problem (1.1) can be substituted by the following operator inclusion:

$$Ax \in B(x, \mu), \tag{6.2}$$

where $A : W_T^{\beta,2}(I, \mathbb{R}^n) \to L_2(I, \mathbb{R}^n)$, $\quad Ax = D_{0^+}^{\beta} x$. From Corollary 4.4

$$KerA \cong \mathbb{R}^n \cong CokerA,$$

and A is a Fredholm operator of index zero. The projection

$$\pi_A : L_2(I, \mathbb{R}^n) \to \mathbb{R}^n,$$

is defined as

$$\pi_A(g) = \frac{1}{T} \int_0^T \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s) ds,$$

and the homeomorphism $l_A : \mathbb{R}^n \to \mathbb{R}^n$ is an identity operator. The space $L_2(I, \mathbb{R}^n)$ can be represented as

$$L_2(I, \mathbb{R}^n) = \mathfrak{L}_0 \oplus \mathfrak{L}_1,$$

where $\mathfrak{L}_0 = CokerA$ and $\mathfrak{L}_1 = ImA$. The decomposition of an element $g \in L_2(I, \mathbb{R}^n)$ is denoted by

$$g = g(0) + g(1), \qquad g(0) \in \mathfrak{L}_0, \quad g(1) \in \mathfrak{L}_1.$$

So inclusion (6.2) is equivalent to the following inclusion:

$$x \in H(x, \mu), \tag{6.3}$$

where $H : C(I, \mathbb{R}^n) \times \mathbb{R} \to K(C(I, \mathbb{R}^n))$,

$$H(x, \mu) = Px + (\pi_A + k_A) \circ B(x, \mu).$$

It is clear that H is a J^c -multimap. In order to show that H is completely u.s.c., let $\Omega \subset C(I, \mathbb{R}^n) \times \mathbb{R}$ be a bounded subset and $(x, \mu) \in \Omega$. Taking arbitrarily $\gamma \in Q(\Omega)$, then there exist $u \in \Pi(x, \mu)$, such that

$$\gamma(t) = \mu a x(t) + f(t, x(t), u(t), \mu), \quad \text{for a.e. } t \in I.$$

From $(f1)$, it follows that

$$|\gamma(t)| \leq |x(t)|((a+c)\mu + cu(t)), \quad \text{for a.e. } t \in I. \tag{6.4}$$

Since $u \in \Pi(x, \mu)$, there is $g \in L_1(I, \mathbb{R}^m)$, such that

$$g(t) \in G(t, x(t), u(t), \mu) \text{ for a.e. } t \in I,$$

and

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s) ds.$$

From $(g3)$ we have

$$\begin{aligned} |u(t)| \leq & \int_0^t |(t-s)^{\alpha-1} g(s)| ds \leq \tfrac{T^{\alpha-1}}{\Gamma(\alpha)} \int_0^t B(|x(s)| + |u(s)| + |\mu|) ds \\ & \leq \tfrac{T^{\alpha-1}}{\Gamma(\alpha)} B\sqrt{T} ||x||_2 + \int_0^t B|u(s)| ds + \beta T|\mu|. \end{aligned}$$

From Lemma 6.1, it follows that

$$|u(t)| \leq \frac{T^{\alpha-1}}{\Gamma(\alpha)}(B\sqrt{T}||x||_2 + BT|\mu|)e^{BT}, \quad \text{for all} t \in I. \tag{6.5}$$

From (6.5) and (6.4), there exists $M_\Omega > 0$ such that $|\gamma(t)| < M_\Omega$, for a.e. $t \in I$, i.e., the set $Q(\Omega)$ is bounded in $L_2(I, \mathbb{R}^n)$. Notice that the operator

$$\pi_A + k_A : L_2(I, \mathbb{R}^n) \to W^{1,2}(I, \mathbb{R}^n),$$

is continuous and the map P takes values in $\mathbb{R}^n$. Then, the set $H(\Omega)$ is bounded in $W^{1,2}(I, \mathbb{R}^n)$, and hence, it is a relative compact set in $C(I, \mathbb{R}^n)$. So, H is a completely u.s.c. J^c-multimap.
Now we want to show that for each $\mu \neq 0$, there exists $\delta_\mu > 0$ such that inclusion (6.3) has no non-trivial solution on $B_C(0, \delta_\mu) \times \{\mu\}$. In fact, to contrary assume that $(x, \mu) \in B_C(0, \delta_\mu) \times \{\mu\}$ is a nontrivial solution of (6.3). Then there is $u \in \Pi(x, \mu)$ such that

$$D^\beta_{0+}x(t) = \mu a x(t) + f(t, x(t), u(t), \mu), \quad \text{for a.e. } t \in I. \tag{6.6}$$

Therefore

$$\int_0^T \langle \mu x(t), \mu a x(t) + f(t, x(t), u(t), \mu)\rangle dt = \int_0^T \langle \mu x(t), D^\beta_{0+}x(t)\rangle dt,$$

since D^β_{0+} is continuous on $x(t)$ for all $t \in I$ and $x(T) = x(0)$, one can set $u = x(t)$ and $dv = D^\beta_{0+}x(t)dt$. From integration by parts we have

$$\int_0^T \langle \mu x(t), D^\beta_{0+}x(t)\rangle dt = 0. \tag{6.7}$$

On the other hand

$$\begin{aligned}
&\textstyle\int_0^T \langle \mu x(t), \mu a x(t) + f(t, x(t), u(t), \mu)\rangle dt \\
&\quad\geq a\mu^2||x||_2^2 - |\mu| \textstyle\int_0^T |x(t)||f(t, x(t), u(t), \mu)dt \\
&\quad\geq a\mu^2||x||_2^2 - c|\mu|\frac{T^{B-1}}{\Gamma(\alpha)} \textstyle\int_0^T x^2(t)(|\mu| + |u(t)|)dt \\
&\quad\geq (a-c)\mu^2||x||_2^2 - c|\mu| \textstyle\int_0^T x^2(t)(B\sqrt{T}||x||_2 + BT|\mu|)e^{BT}dt \\
&\quad= (a - c - \frac{cT^\alpha Be^{TB}}{\Gamma(\alpha)})\mu^2||x||_2^2 - \frac{c|\mu|T^{\alpha-\frac{1}{2}}Be^{TB}||x||_2^3}{\Gamma(\alpha)} \\
&\quad> 0,
\end{aligned} \tag{6.8}$$

provided

$$0 < ||x||_2 < \frac{(aT - c - \frac{cT^\alpha Be^{TB}}{\Gamma(\alpha)})\mu^2}{\frac{c|\mu|T^{\alpha-\frac{1}{2}}Be^{T\beta}}{\Gamma(\alpha)}}. \tag{6.9}$$

Therefore, inclusion (6.3) has no solution (x, μ) that satisfies (6.9). Thus, for sufficiently small δ_μ, we obtain a contradiction.

Now, for evaluating the topological degree

$$deg(i - H(., \mu), B_C(0, \delta_\mu)),$$

for a given $\mu \neq 0$, let us consider the multimap

$$\begin{aligned}
\Sigma_\mu &: B_C(0, \delta_\mu) \times [0, 1] \to K(C(I, \mathbb{R}^n)), \\
\Sigma_\mu &\ (x, \lambda) = Px + (\pi_A + k_A) \circ \varphi(B(x, \mu), \lambda),
\end{aligned}$$

where $\varphi : L_2((I, \mathbb{R}^n) \times [0, 1] \to L_2((I, \mathbb{R}^n),$

$$\varphi(g, \lambda) = g_{(0)} + \lambda g_{(1)}, \quad g_{(0)} \in \mathfrak{L}_0, \quad g_{(1)} \in \mathfrak{L}_1.$$

It is clear that Σ_μ is a compact J^c-multimap. Assume that there is $(x_*, \lambda_*) \in \partial B_C(0, \delta_\mu) \times [0, 1]$ such that $x_* \in \Sigma_\mu(x_*, \lambda_*)$. Then there are $\gamma^* \in L_2(I, \mathbb{R}^n)$ and $u^* \in \Pi(x_*, \mu)$, such that

$$\gamma^*(t) = a\mu x_*(t) + f(t, x_*(t), u^*(t), \mu), \quad \text{for a.e.} t \in I,$$

and

$$\begin{cases} D^\beta x_*(t) = \lambda_* \gamma^*(1), \\ \\ 0 = \gamma^*(0), \end{cases} \tag{6.10}$$

where $\gamma^*_{(0)} + \gamma^*_{(1)} = \gamma^*, \gamma^*_{(0)} \in \mathfrak{L}_0, \gamma^*_{(1)} \in \mathfrak{L}_1$.
If $\lambda_* \neq 0$, then

$$\int_0^T \langle \mu x_*(t), \gamma^*(t) \rangle dt = \frac{1}{\lambda_*} \int_0^T \langle \mu x_*(t), D^\beta x_*(t) \rangle dt = 0.$$

On the other hand, from $||x_*||_2 \leq \sqrt{T}||x_*||_C = \sqrt{T}\delta_\mu$, it follows that x_* satisfies relation (6.9) for sufficiently small δ_μ. Therefore,

$$\int_0^T \langle \mu x_*(t), \gamma^*(t) \rangle dt > 0,$$

giving a contradiction.

If $\lambda_* = 0$, then $x_* \in KerA$ i.e. $x_*(t) = w \in \mathbb{R}^n$ for all $t \in I$. Since the fact that w satisfies relation (6.9), we have

$$\int_0^T \langle \mu w, \gamma(t) \rangle dt > 0$$

for all $u \in \Pi(w, \mu)$, where

$$\gamma(t) = a\mu w + f(t, w, u(t), \mu) \in B(w, \mu), \quad \text{for a.e.} \quad t \in I.$$

Notice that

$$\int_0^T \langle \mu w, \gamma(t) \rangle dt = T\langle \mu w, \pi_A \gamma \rangle.$$

Consequently,

$$\langle \mu w, \pi_A \gamma \rangle > 0, \quad \text{for all } \gamma \in B(w, \mu). \tag{6.11}$$

In particular,

$$0 < \langle \mu w, \pi_A \gamma^* \rangle = \langle \mu w, \pi_A \gamma^*_{(0)} \rangle = 0,$$

Which is a contradiction.
So, Σ_μ is a homotopy connecting the multimaps $\Sigma_\mu(., 1) = H(., \mu)$ and

$$\Sigma_\mu(., 0) = P + \pi_A B(., \mu).$$

According to the invariant property of the topological degree, we have

$$deg(i - H(., \mu), B_C(0, \delta_\mu)) = deg(i - P - \pi_A B(., \mu), B_C(0, \delta_\mu)).$$

Notice that the multimap $P + \pi_A B(., \mu)$ takes values in $\mathbb{R}^n$, and hence

$$deg(i - P - \pi_A B(., \mu), B_C(0, \delta_\mu)) = deg(i - P - \pi_A B(., \mu), B_{\mathbb{R}^n}(0, \delta_\mu)).$$

In the space $\mathbb{R}^n$, the vector field $i - P - \pi_A B(., \mu)$ has the form

$$i - P - \pi_A B(., \mu) = -\pi_A B(., \mu).$$

From (6.11), it follows that $\pi_A B(., \mu)$ and μi are homotopic on $\partial B_{\mathbb{R}^n}(0, \delta_\mu)$. So, we obtain

$$deg(-\pi_A B(., \mu), B_{\mathbb{R}^n}(0, \delta_\mu)) = deg(-\mu i, B_{\mathbb{R}^n}(0, \delta_\mu)) = -sign(\mu).$$

Thus, the bifurcation index $Bi[H; (0,0)] = -2$. From (6.9)-(6.8), it follows that $(0,0)$ is the unique bifurcation point of system (1.1). To complete the proof, we need only to apply Theorem (5.1) with a remark that the case (b) of Theorem 1 could not appear.

□

Remark 6.3. By the last theorem, it is shown that not only there is a solution set for the mentioned control feedback system (1.1), but also the global structure of nontrivial solution set of such systems like being connected and bifurcating from $(0,0)$ and tending to infinity is obtained.

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