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The Study of manageable functions and approximate fixed point property with their application

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Abstract

In this work, we want to investigate and improve the concept of manageable functions and $\widehat{Man}(\mathbf{R})$-contractions in order to extend the theory of multi-valued contractions and fixed point results.

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1. Introduction

In 1922, Banach established the most famous fundamental fixed-point theorem (so-called the Banach contraction principle [1]) which has played an important role in various fields of applied mathematical analysis. It is known that the Banach contraction principle has been extended in many various directions by several authors(see [2, 3, 4, 6, 7, 8]). An interesting direction of research is the extension of the Banach contraction principle of multi-valued maps, known as Nadler's fixed-point theorem [9], Mizoguchi-Takahashi's fixed-point theorem [10], M. Berinde and V. Berinde [3], Ćirić [4], Reich [5], Daffer and Kaneko [6], Rhoades [11], Amini-Harandi [1, 8], Moradi and Khojasteh [12], Du [7] and references therein. Let us recall some basic notations needed in this paper.
Let (X, d) be a metric space. For each $x \in X$ and $A \subseteq X$, let $d(x, A) = \inf_{y \in A} d(x, y)$. Denote by $\mathcal{N}(X)$ the class of all nonempty subsets of X, $\mathcal{C}(X)$ the family of all nonempty closed subsets of X and $\mathcal{CB}(X)$ the family of all nonempty closed and bounded subsets of X. A function $\mathcal{H} : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0, \infty)$ defined by

$$\mathcal{H}(A, B) = max \left\{ \sup_{x \in B} D(x, A), \sup_{x \in A} D(x, B) \right\},$$

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is said to be the Hausdorff metric on $\mathcal{CB}(X)$ induced by d on X, where $D(x,A) = \inf\{d(x,y) : y \in A\}$ for each $A \in \mathcal{CB}(X)$. A point v in X is a fixed-point of a map T if $v = Tv$ (when $T : X \to X$ is a single-valued map) or $v \in Tv$ (when $T : X \to \mathcal{N}(X)$ is a multi-valued map). The set of fixed points of T is denoted by $\mathcal{F}(T)$. Throughout this paper, R and N, denote the set of real and natural numbers, respectively.

Very recently, Du and Khojasteh [13] introduced the notion of *manageable function* as follows:

Definition 1.1. A function $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called *manageable* if the following conditions hold:

(η_1) $\eta(t,s) < s - t$ for all $s,t > 0$.

(η_2) For any bounded sequence $\{t_n\} \subset (0,\infty)$ and any nonincreasing sequence $\{s_n\} \subset (0,\infty)$, it holds

$$\limsup_{n\to\infty} \frac{t_n + \eta(t_n, s_n)}{s_n} < 1.$$

We denote the set of all manageable functions by $\widehat{\mathrm{Man}(\mathbb{R})}$.

In that paper, the authors announced the following result.

Theorem 1.2 (Du and Khojasteh [13], Theorem 10)**.** *Let (X,d) be a metric space, $T : X \to \mathcal{N}(X)$ be a α-admissible multivalued map and $\eta \in \widehat{Man(\mathbb{R})}$. Let*

$$\Omega = \{(\alpha(x,y)d(y,Ty), d(x,y)) \in [0,+\infty) \times [0,+\infty) : x \in X \text{ and } y \in Tx\}.$$

If $\eta(t,s) \geq 0$ for all $(t,s) \in \Omega$ and there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$, then the following statements hold:

(a) *There exists a Cauchy sequence $\{w_n\}_{n\in\mathbb{N}}$ in X such that:*

- **(i)** $w_{n+1} \in Tw_n$ *for all* $n \in \mathbb{N}$,
- **(ii)** $\alpha(w_n, w_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$,
- **(iii)** $\lim_{n\to\infty} d(w_n, w_{n+1}) = \inf_{n\in\mathbb{N}} d(w_n, w_{n+1}) = 0$.

(b) $\inf_{x\in X} d(x,Tx) = 0$*; that is T has the approximate fixed point property on X.*

Definition 1.3. Given a metric space (X,d), we say that a multivalued mapping $T : X \to \mathcal{CB}(X)$ is *continuous* if $\{Tx_n\} \xrightarrow{\mathcal{H}} Tz$ for all sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \xrightarrow{d} z \in X$.

2. Main Result

In order to extend the results of [13], we present the following statement, including its corresponding proof (which is very similar to the proof of Theorem 1.2 in [13]).

Let denote by $\mathcal{C}(X)$ the family of all nonempty, closed subsets of (X,d) (notice that $\mathcal{CB}(X)$ is necessary for defining the metric $\mathcal{H} : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0,\infty)$).

Theorem 2.1. *Let (X,d) be a metric space, $T : X \to \mathcal{C}(X)$ be an α-admissible multi-valued map and $\eta \in \widehat{Man(\mathbb{R})}$. Let*

$$\Omega = \left\{ \Big(\alpha(x,y)\, d(y,Ty),\ d(x,y)\Big) \in (0,\infty) \times (0,\infty) \,\middle|\, \begin{array}{c} x \in X, \quad y \in Tx, \quad x \neq y \\ d(y,Ty) > 0 \text{ and } \alpha(x,y) \geq 1 \end{array} \right\}.$$

If $\eta(t,s) \geq 0$ for all $(t,s) \in \Omega$ and there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$, then the following statements hold:

(a) *There exists a Cauchy sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* X *such that:*

(i) $x_{n+1} \in Tx_n$ *for all* $n \in \mathbb{N}$,

(ii) $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$,

(iii) $\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0$.

(b) $\inf_{x\in X} d(x, Tx) = 0$*; that is* T *has the approximate fixed point property on* X.

Remark 2.2. **(1)** Notice that, since the set Ω is smaller than in Theorem 1.2, then the contractivity condition "$\eta \geq 0$ in Ω" is weaker.

(2) We point out that the set Ω could be empty but, in this case, we will prove that $\{x_0, x_1\}$ contains a fixed point of T.

Proof. First of all, we prove the following two claims:

Claim 1. If there exists $x \in X$ *such that* $x \in Tx$ *and* $\alpha(x, x) \geq 1$, *then conclusions (a) and (b) hold and* x *is a fixed point of* T *verifying* $\alpha(x, x) \geq 1$. It follows by taking $x_n = x$ for all $n \in \mathbb{N}$.

Claim 2. If there exist $x, y \in X$ *such that* $y \in Tx \cap Ty$ *and* $\alpha(x, y) \geq 1$, *then conclusions (a) and (b) hold and* y *is a fixed point of* T *verifying* $\alpha(y, y) \geq 1$. In this case, consider the sequence $\{x_n\}_{n\in\mathbb{N}}$ *in* X defined by $x_1 = x$ and $x_n = y$ for all $n \geq 2$. As T is α-admissible, $\alpha(x, y) \geq 1$ and $y \in Ty$, we obtain $\alpha(y, y) \geq 1$. Then Claim 1 is applicable to y.

Let $x_0 \in X$ and $x_1 \in Tx_0$ be the points (guaranteed by hypothesis) such that $\alpha(x_0, x_1) \geq 1$.

- If $x_1 = x_0$, then $x_0 \in Tx_0$, and it follows from Claim 1 that conclusions (a) and (b) hold and x_0 is a fixed point of T verifying $\alpha(x_0, x_0) \geq 1$.
- If $x_1 \in Tx_1$, then the proof is finished by Claim 2 using $x = x_0$ and $y = x_1$.

In the previous cases, Ω could be empty (but the proof is finished). Next, assume that $x_1 \neq x_0$ and $x_1 \notin Tx_1$. Since $Tx_1 \in \mathcal{C}(X)$, then we deduce that,

$$d(x_0, x_1) > 0 \quad \text{and} \quad d(x_1, Tx_1) > 0.$$

Since $\alpha(x_0, x_1) \geq 1$, it follows that,

$$\Big(\alpha(x_0, x_1)\, d(x_1, Tx_1),\ d(x_0, x_1)\Big) \in \Omega, \tag{2.1}$$

which means that Ω is not empty. In this case, let define $\lambda : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by,

$$\lambda(t, s) = \begin{cases} \dfrac{t + \eta(t, s)}{s}, & \text{if } (t, s) \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

By (η_1) and the fact that $t > 0$ and $s > 0$ for all $(t, s) \in \Omega$, we know that,

$$0 < \lambda(t, s) < 1 \quad \text{for all } (t, s) \in \Omega. \tag{2.2}$$

Since $\eta \in \widehat{\mathrm{Man}(\mathbb{R})}$ and $\eta(t, s) \geq 0$ for all $(t, s) \in \Omega$, we have that,

$$0 < t \leq s\, \lambda(t, s) \quad \text{for all } (t, s) \in \Omega. \tag{2.3}$$

By (2.1) and (2.2), we have,

$$0 < \lambda(\alpha(x_0, x_1) d(x_1, Tx_1), d(x_0, x_1)) < 1.$$

Take

$$\varepsilon_1 = \left(\frac{\alpha(x_0, x_1)}{\sqrt{\lambda(\alpha(x_0, x_1)d(x_1, Tx_1), d(x_0, x_1))}} - 1 \right) d(x_1, Tx_1).$$

Then $\varepsilon_1 > 0$. Since

$$\begin{aligned} d(x_1, Tx_1) &< d(x_1, Tx_1) + \varepsilon_1 \\ &= \frac{\alpha(x_0, x_1)}{\sqrt{\lambda(\alpha(x_0, x_1)d(x_1, Tx_1), d(x_0, x_1))}} d(x_1, Tx_1), \end{aligned}$$

and $d(x_1, Tx_1)$ is an infimum, there exists $x_2 \in Tx_1$ such that,

$$d(x_1, x_2) < \frac{\alpha(x_0, x_1)}{\sqrt{\lambda(\alpha(x_0, x_1)d(x_1, Tx_1), d(x_0, x_1))}} d(x_1, Tx_1).$$

Since T is α-admissible, $\alpha(x_0, x_1) \geq 1$ and $x_2 \in Tx_1$, we obtain $\alpha(x_1, x_2) \geq 1$.

- If $x_2 = x_1$, then Claim 1 guarantees that conclusions (a) and (b) hold, and x_1 is a fixed point of T verifying $\alpha(x_1, x_1) \geq 1$.
- If $x_2 \in Tx_2$, then the proof is finished by Claim 2 using $x = x_1$ and $y = x_2$.

On the contrary, assume that $x_2 \neq x_1$ and $x_2 \notin Tx_2$. Therefore $d(x_1, x_2) > 0$ and $d(x_2, Tx_2) > 0$. By taking

$$\varepsilon_2 = \left(\frac{\alpha(x_1, x_2)}{\sqrt{\lambda(\alpha(x_1, x_2)d(x_2, Tx_2), d(x_1, x_2))}} - 1 \right) d(x_2, Tx_2) > 0,$$

and taking into account that,

$$\begin{aligned} d(x_2, Tx_2) &< d(x_2, Tx_2) + \varepsilon_1 \\ &= \frac{\alpha(x_1, x_2)}{\sqrt{\lambda(\alpha(x_1, x_2)d(x_2, Tx_2), d(x_1, x_2))}} d(x_2, Tx_2), \end{aligned}$$

there exists $x_3 \in Tx_2$ such that,

$$d(x_2, x_3) < \frac{\alpha(x_1, x_2)}{\sqrt{\lambda(\alpha(x_1, x_2)d(x_2, Tx_2), d(x_1, x_2))}} d(x_2, Tx_2).$$

Since T is α-admissible, $\alpha(x_1, x_2) \geq 1$ and $x_3 \in Tx_2$, we obtain $\alpha(x_2, x_3) \geq 1$. The cases $x_3 = x_2$ or $x_3 \in Tx_3$ immediately finish the proof by using Claims 1 or 2. On the contrary, we continue assuming that $d(x_2, x_3) > 0$ and $d(x_3, Tx_3) > 0$.

By repeating the previous process again and again, we construct recursively a sequence $\{x_n\}$ such that $x_{n+1} \in Tx_n$ and $\alpha(x_{n-1}, x_n) \geq 1$. It is possible that we can find $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$ or $x_{n_0+1} \in Tx_{n_0+1}$. In these cases, Claims 1 and 2 finish the proof and we conclude that T has a fixed point. On the contrary case, if this process never ends, we can consider a sequence $\{x_n\}_{n \in \mathbb{N}}$ in X satisfying:

$$x_n \in Tx_{n-1}, \quad d(x_{n-1}, x_n) > 0, \quad d(x_n, Tx_n) > 0, \quad \alpha(x_{n-1}, x_n) \geq 1 \quad \text{and}$$

$$d(x_n, x_{n+1}) < \frac{\alpha(x_{n-1}, x_n)}{\sqrt{\lambda(\alpha(x_{n-1}, x_n)d(x_n, Tx_n), d(x_{n-1}, x_n))}} d(x_n, Tx_n). \tag{2.4}$$

for each $n \in \mathbb{N}$.

It follows that,

$$\Big(\alpha(x_{n-1}, x_n)\, d(x_n, Tx_n),\ d(x_{n-1}, x_n)\Big) \in \Omega \quad \text{for all } n \in \mathbb{N}.$$

By (2.3), we have,

$$\alpha(x_{n-1},x_n)\,d(x_n,Tx_n)\le d(x_{n-1},x_n)\,\lambda(\alpha(x_{n-1},x_n)\,d(x_n,Tx_n),d(x_{n-1},x_n))\quad\text{for each } n\in\mathbb{N}. \tag{2.5}$$

Hence, for each $n \in \mathbb{N}$, by combining (2.4) and (2.5), we get,

$$d(x_n,x_{n+1})<\sqrt{\lambda(\alpha(x_{n-1},x_n)\,d(x_n,Tx_n),d(x_{n-1},x_n))}\;d(x_{n-1},x_n) \tag{2.6}$$

which means that the sequence $\{d(x_{n-1},x_n)\}_{n\in\mathbb{N}}$ is strictly decreasing in $(0,\infty)$. So,

$$\gamma=\lim_{n\to\infty}d(x_n,x_{n+1})=\inf_{n\in\mathbb{N}}d(x_n,x_{n+1})\ge 0\quad\text{exists.}$$

By (2.5), we have,

$$\alpha(x_{n-1},x_n)\,d(x_n,Tx_n)\le d(x_{n-1},x_n)\quad\text{for all } n\in\mathbb{N},$$

which means that $\{\alpha(x_{n-1},x_n)\,d(x_n,Tx_n)\}_{n\in\mathbb{N}}$ is a bounded sequence. By (η_2), we have that,

$$\limsup_{n\to\infty}\lambda(\alpha(x_{n-1},x_n)\,d(x_n,Tx_n),d(x_{n-1},x_n))<1. \tag{2.7}$$

Now, we claim $\gamma = 0$. To prove it, suppose that $\gamma > 0$. Then, by (2.7) and taking limit superior in (2.6), we get,

$$\gamma\le\sqrt{\limsup_{n\to\infty}\lambda(\alpha(x_{n-1},x_n)d(x_n,Tx_n),d(x_{n-1},x_n))}\;\gamma<\gamma,$$

which is a contradiction. Hence we deduce that,

$$\lim_{n\to\infty}d(x_n,x_{n+1})=\inf_{n\in\mathbb{N}}d(x_n,x_{n+1})=0. \tag{2.8}$$

To complete the proof of (a), it is sufficient to show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. For each $n \in \mathbb{N}$, let

$$\rho_n=\sqrt{\lambda(\alpha(x_{n-1},x_n)\,d(x_n,Tx_n),d(x_{n-1},x_n))}.$$

Then $\rho_n \in (0,1)$ for all $n \in \mathbb{N}$. By (2.6), we obtain,

$$d(x_n,x_{n+1})<\rho_n\,d(x_{n-1},x_n)\quad\text{for all } n\in\mathbb{N}. \tag{2.9}$$

From (2.7), we have $\limsup_{n\to\infty}\rho_n < 1$, so there exists $c \in [0,1)$ and $n_0 \in \mathbb{N}$, such that,

$$\rho_n\le c\quad\text{for all } n\in\mathbb{N}\text{ with } n\ge n_0. \tag{2.10}$$

For any $n \ge n_0$, since $\rho_n \in (0,1)$ for all $n \in \mathbb{N}$ and $c \in [0,1)$, taking into account (2.9) and (2.10) conclude that,

$$d(x_n,x_{n+1})<\rho_n\,d(x_{n-1},x_n)<\ldots<\rho_n\rho_{n-1}\rho_{n-2}\cdot\ldots\cdot\rho_{n_0}\,d(x_0,x_1)\le c^{n-n_0+1}d(x_0,x_1).$$

Put $\alpha_n=\frac{c^{n-n_0+1}}{1-c}d(x_0,x_1)$, $n \in \mathbb{N}$. For $m,n \in \mathbb{N}$ with $m > n \ge n_0$, we have from the last inequality that

$$d(x_n,x_m)\le\sum_{j=n}^{m-1}d(x_j,x_{j+1})<\alpha_n.$$

Since $c \in [0,1)$, $\lim_{n\to\infty}\alpha_n = 0$ and therefore,

$$\limsup_{n\to\infty}\;\{d(x_n,x_m):m>n\}=0.$$

As a consequence, $\{x_n\}$ is a Cauchy sequence in X. Let $w_n = x_{n-1}$ for all $n \in \mathbb{N}$. Then $\{w_n\}_{n\in\mathbb{N}}$ is the desired Cauchy sequence in (a).

To see (b), since $x_n \in Tx_{n-1}$ for each $n \in \mathbb{N}$, we have,

$$\inf_{x\in X}d(x,Tx)\le d(x_n,Tx_n)\le d(x_n,x_{n+1})\quad\text{for all } n\in\mathbb{N}. \tag{2.11}$$

Combining (2.8) and (2.11) yields,

$$\inf_{x\in X}d(x,Tx)=0.$$

This completes the proof. □

2.1. Existence of fixed points under right-continuity

Definition 2.3. Let (X, d) be a metric space, let $\alpha : X \times X \to \mathbb{R}$ be a function and let $T : X \to \mathcal{CB}(X)$ be a multivalued mapping. We say that T is *α-right-continuous* if $\{Tx_n\} \xrightarrow{\mathcal{H}} Tz$ for all sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \xrightarrow{d} z \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.

Obviously, every continuous multi-valued mapping is α-right-continuous when $\alpha = 1$. In the following result, we use that,

$$d(a, B) \leq \mathcal{H}(A, B) \quad \text{for all } A, B \in \mathcal{CB}(X) \text{ and all } a \in A.$$

Theorem 2.4. *Under the hypothesis of Theorem 2.1, additionally assume that (X, d) is complete and $T : X \to \mathcal{CB}(X)$ is α-right-continuous. Then the sequence $\{x_n\}$ converges to a fixed point of T.*

Proof. Since $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d), there exists $\omega \in X$ such that $\{x_n\} \to \omega$. As T is α-right-continuous, we deduce that $\{Tx_n\} \xrightarrow{\mathcal{H}} T\omega$. In particular, as $d(x_n, T\omega) \leq \mathcal{H}(Tx_{n+1}, T\omega)$ for all $n \in \mathbb{N}$, we conclude that $\omega \in \overline{T\omega} = T\omega$, so ω is a fixed point of T. □

2.2. A weaker appropriate condition

Definition 2.5. Let (X, d) be a metric space, let $\{x_n\} \subseteq X$ be a sequence, let $\alpha : X \times X \to \mathbb{R}$ be a function and let $T : X \to \mathcal{N}(X)$ be a multivalued mapping. We will say that $\{x_n\}$ is an *Picard sequence of* (T, α) if for all $n \in \mathbb{N}$,

$$x_{n+1} \in Tx_n, \quad x_n \neq x_{n+1} \quad \text{and} \quad \alpha(x_n, x_{n+1}) \geq 1.$$

Let consider the following property.

($\mathcal{A}$) For all $\varepsilon > 0$, all $m \in \mathbb{N}$ and all convergent Picard sequence $\{x_n\}_{n\in\mathbb{N}}$ of (T, α), there exists $k \in \mathbb{N}$ such that $k > m$ and $d(x_k, T\omega) < \varepsilon$ (where $\omega = \lim_{n\to\infty} x_n \in X$).

In the following result, we do not assume that the range of T is included in $\mathcal{CB}(X)$ but in $\mathcal{C}(X)$.

Theorem 2.6. *Under the hypothesis of Theorem 2.1, additionally assume that (X, d) is complete and property $(\mathcal{A})$ is satisfied. Then the sequence $\{x_n\}$ converges to a fixed point of T.*

Proof. Since $\{x_n\}$ is a Cauchy sequence in the complete space (X, d), there exists $\omega \in X$ such that $\{x_n\} \to \omega$. By Theorem 2.1, $\{x_n\}$ is a Picard sequence of (T, α). By property $(\mathcal{A})$ using $\{\varepsilon_k = 1/k > 0\}_{k\in\mathbb{N}}$, there exists a subsequence $\{x_{n(k)}\}_{k\in\mathbb{N}}$ of $\{x_n\}$ such that $\{d(x_{n(k)}, T\omega)\} \to 0$. As $\{x_n\}$ converges to ω, then $\{x_{n(k)}\}_{k\in\mathbb{N}}$ also converges to ω, so $\omega \in \overline{T\omega} = T\omega$ and ω is a fixed point of T □

Next, we show that Theorem 2.6 improves Theorem 2.4. Notice that the range of T must be included in $\mathcal{CB}(X)$ is order to use the Hausdorff metric $\mathcal{H}$.

Lemma 2.7. *Let (X, d) be a metric, let $\alpha : X \times X \to \mathbb{R}$ be a function and let $T : X \to \mathcal{CB}(X)$ be a multivalued mapping. If T is α-right-continuous, then property $(\mathcal{A})$ is satisfied.*

Proof. Let $\{x_n\}$ be an arbitrary Picard sequence of (T, α) such that $\{x_n\} \to \omega \in X$. Since $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, the α-right-continuity of T implies that $\{Tx_n\} \xrightarrow{\mathcal{H}} Tz$. In particular, $d(x_{n+1}, T\omega) \leq \mathcal{H}(Tx_n, T\omega)$ for all $n \in \mathbb{N}$. As a consequence, $\{d(x_{n+1}, T\omega)\} \to 0$. Therefore, for all $\varepsilon > 0$ and all $m \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $k > m$ and $d(x_k, T\omega) < \varepsilon$. Hence, property $(\mathcal{A})$ is satisfied. □

2.3. Existence of fixed points under regularity

Definition 2.8. Let (X,d) be a metric space and let $\alpha : X \times X \to \mathbb{R}$ be a function. We will say that (X,d) is *α-regular* if it satisfies the following property:

($\mathcal{R}$) If $\{z_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $\{z_n\} \to \omega \in X$ and $\alpha(z_n, z_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(z_n, \omega) \geq 1$ for all $n \in \mathbb{N}$.

In the following result, we guarantee that T has a fixed point.

Theorem 2.9. *Let (X,d) be a complete metric space, let $\alpha : X \times X \to \mathbb{R}$ be a function, let $T : X \to \mathcal{C}(X)$ be an α-admissible multivalued map and let $\eta \in \widehat{Man(\mathbb{R})}$ be a manageable function. If $\eta(t,s) \geq 0$ for all*

$$(t,s) \in \left\{ \Big(\alpha(x,y)\, d(y,Ty),\ d(x,y)\Big) \in (0,\infty) \times (0,\infty) \,\middle|\, \begin{array}{c} x,y \in X, \quad x \neq y \\ d(y,Ty) > 0 \text{ and } \alpha(x,y) \geq 1 \end{array} \right\}, \tag{2.12}$$

(X,d) is α-regular and there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$, then T has at least a fixed point (in fact, the sequence guaranteed by Theorem 2.1 converges to a fixed point of T).

Proof. Let Ω' be the subset of $(0,\infty) \times (0,\infty)$ given by (2.12). Notice that $\Omega \subseteq \Omega'$, where Ω is given in Theorem 2.1. Therefore, the condition "$\eta(t,s) \geq 0$ for all $(t,s) \in \Omega'$" implies that "$\eta(t,s) \geq 0$ for all $(t,s) \in \Omega$". Hence, Theorem 2.1 is applicable, and its proof can be repeated here point by point.

Notice that, if we can apply Claim 1 or Claim 2, the considered sequence $\{x_n\}_{n\geq 2}$ is constant and its limit is a fixed point of T. In this case, the proof is finished. Suppose that the process to consider the sequence $\{x_n\}$ is not finite. In such a case, we have proved that Ω is not empty (so Ω' is not empty) and the sequence $\{x_n\}$ satisfies, for all $n \in \mathbb{N}$,

$$x_n \in Tx_{n-1}, \quad d(x_{n-1},x_n) > 0, \quad d(x_n,Tx_n) > 0, \quad \alpha(x_{n-1},x_n) \geq 1 \quad \text{and}$$

$$d(x_n,x_{n+1}) < \frac{\alpha(x_{n-1},x_n)}{\sqrt{\lambda(\alpha(x_{n-1},x_n)d(x_n,Tx_n), d(x_{n-1},x_n))}}\, d(x_n,Tx_n),$$

where λ is now defined replacing Ω by Ω'.

Since $\{x_n\}$ is a Cauchy sequence in the complete space (X,d), there exists $\omega \in X$ such that $\{x_n\} \to \omega$. As (X,d) is α-regular, we deduce that,

$$\alpha(x_n,\omega) \geq 1 \quad \text{for all } n \in \mathbb{N}.$$

Consider the set $S = \{n \in \mathbb{N} : x_n = \omega\}$. We distinguish two cases.

Case 1. S is not finite. In this case, $x_{n-1} \in Tx_n = T\omega$ for all $n \in S$, that is $\{x_n\}$ cointains a subsequence $\{x_{n(k)}\}_{k\in\mathbb{N}}$ such that $x_{n(k)} \in T\omega$ for all $k \in \mathbb{N}$. As $\{x_n\}$ converges to ω, then $\{x_{n(k)}\}_{k\in\mathbb{N}}$ also converges to ω, so $\omega \in \overline{T\omega} = T\omega$ and ω is a fixed point of T.

Case 2. S is finite. In this case, there exists $n_0 \in \mathbb{N}$ such that $x_n \neq \omega$ for all $n \geq n_0$. We are also going to show that $\omega \in T\omega$ reasoning by contradiction. Assume that $\omega \notin T\omega$, that is, $d(\omega,T\omega) > 0$. In this case,

$$\Big(\alpha(x_n,\omega)\, d(\omega,T\omega),\ d(x_n,\omega)\Big) \in \Omega' \quad \text{for all } n \geq n_0.$$

In particular, for all $n \geq n_0$,

$$d(\omega,T\omega) \leq \alpha(x_n,\omega)\, d(\omega,T\omega) \leq d(x_n,\omega)\, \lambda(\alpha(x_n,\omega)\, d(\omega,T\omega), d(x_n,\omega)) \leq d(x_n,\omega).$$

Letting $n \to \infty$, we deduce that $d(\omega,T\omega) = 0$, which contradicts the fact that $d(\omega,T\omega) > 0$. As a consequence, necessarily $\omega \in T\omega$. □

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