Convergence of general composite iterative method for infinite family of nonexpansive mappings in Hilbert spaces

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Abstract

In this paper by using $W_n$-mapping, we introduce a composite iterative method for finding a common fixed point for infinite family of nonexpansive mappings and a solution of a certain variational inequality. Furthermore, the strong convergence of the proposed iterative method is established. Finally, some simulation examples are presented. Our results improve and extend the previous results. ©2016 All rights reserved.

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1. Introduction

Let $H$ be a real Hilbert space, whose inner product and norm are denoted by \langle , \rangle and $|| . ||$, respectively. Let $C$ be a nonempty closed convex subset of $H$ and $T$ is a nonlinear mapping. We use $F(T)$ to denote the set of fixed points of $T$ (i.e., $F(T) = \{ x \in H : Tx = x \}$). Recall that a self mapping $T$ of $C$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in H$ and is a contraction, if there exists a constant $\alpha \in (0, 1)$ such that $\|Tx - Ty\| \leq \alpha \|x - y\|$ for all $x, y \in C$.

A bounded linear operator $A$ on $H$ is called strongly positive with coefficient $\bar{\gamma} > 0$ if,

$$\langle Ax, x \rangle \geq \bar{\gamma} \| x \|^2, \forall x \in H.$$ 

In 2005, Kim and Xu \cite{4} introduced the following iteration process:

$$x_0 = x \in C \text{ chosen arbitrary },$$
\[ y_n = \beta_n x_n + (1 - \beta_n)T x_n, \]
\[ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n. \quad (1.1) \]

They proved in a uniformly smooth Banach space, the sequence \( \{x_n\} \) defined by (1.1) converges strongly to a fixed point of \( T \). In 2009 Cho and Qin [2] considered the following composite iterative algorithm:

\[ x_0 \in H \text{ chosen arbitrary}, \]
\[ z_n = \gamma_n x_n + (1 - \gamma_n) T x_n, \]
\[ y_n = \beta_n x_n + (1 - \beta_n) T z_n, \]
\[ x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1 - \delta_n) I - \alpha_n A) y_n, \forall \ n \geq 0. \quad (1.2) \]

In 2009 Wangkeeree and Kamraksa [8] introduced a new iterative scheme:

\[ x_0 = x \in C \text{ chosen arbitrary}, \]
\[ z_n = \gamma_n x_n + (1 - \gamma_n) W x_n, \]
\[ y_n = \beta_n x_n + (1 - \beta_n) W z_n, \]
\[ x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + ((1 - \delta_n) I - \alpha_n A) P_C (y_n - \lambda_n B y_n), \quad (1.3) \]

where the mapping \( W_n \) defined by Shimoji and Takahashi [6], as follows:

\[ U_{n,n+1} = I, \]
\[ U_{n,n} = \gamma_n T_n U_{n,n+1} + (1 - \gamma_n) I, \]
\[ U_{n,n-1} = \gamma_{n-1} T_{n-1} U_{n,n} + (1 - \gamma_{n-1}) I, \]
\[ \vdots \]
\[ U_{n,k} = \gamma_k T_k U_{n,k+1} + (1 - \gamma_k) I, \]
\[ U_{n,k-1} = \gamma_{k-1} T_{k-1} U_{n,k} + (1 - \gamma_{k-1}) I, \]
\[ \vdots \]
\[ U_{n,2} = \gamma_2 T_2 U_{n,3} + (1 - \gamma_2) I, \]
\[ W_n = U_{n,1} = \gamma_1 T_1 U_{n,2} + (1 - \gamma_1) I, \quad (1.4) \]

where \( \gamma_1, \gamma_2, \ldots \) are real numbers such that \( 0 \leq \gamma_n \leq 1, T_1, T_2, \ldots \) are an infinite family of mappings of \( H \) into itself, note that the nonexpansivity of each \( T_i \) ensures the nonexpansivity of \( W_n \). In 2010 Singthong and Suantai [7] introduced an iterative method as follows:

\[ x_0 = x \in C \text{ chosen arbitrary}, \]
\[ y_n = \beta_n x_n + (1 - \beta_n) K x_n, \]
\[ x_{n+1} = P_C (\alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n), \quad (1.5) \]

where \( K \)-mapping defined by Kangtunyakarn and Suantai [3] as follows:

\[ U_{n,1} = \lambda_{n,1} T_1 + (1 - \lambda_{n,1}) I, \]
\[ U_{n,2} = \lambda_{n,2} T_2 U_{n,1} + (1 - \lambda_{n,2}) U_{n,1}, \]
\[ U_{n,3} = \lambda_{n,3} T_3 U_{n,2} + (1 - \lambda_{n,3}) U_{n,2}; \]
\[ U_{n,N-1} = \lambda_{n,N-1} T_{n-1} U_{n,N-1} + (1 - \lambda_{n,N}) U_{n,N-1}, \]
\[ K_n = U_{n,N} = \lambda_{n,N} T_N U_{n,N-1} + (1 - \lambda_{n,N}) U_{n,N-1}, \] (1.6)

where \( \{T_i\}_{i=1}^{N} \) are finite family of nonexpansive mappings and the sequences \( \{\lambda_{n,i}\}_{i=1}^{N} \) are in \([0,1]\). The mapping \( K_n \) is called the \( K \)-mapping generated by \( T_1, T_2, \ldots, T_N \) and \( \lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N} \).

Through out this paper inspired by Singthong and Suantai \cite{7} and Wangkeeree and Kumraksa \cite{8}, we introduce a composite iteration method for infinite family of nonexpansive mappings as follows:

\[ x_0 = x \in C \text{ chosen arbitrary}, \]
\[ z_n = \gamma_n x_n + (1 - \gamma_n) W_n x_n, \]
\[ y_n = \beta_n x_n + (1 - \beta_n) W_n z_n, \]
\[ x_{n+1} = P_C [\alpha_n \gamma f(x_n) + \delta_n x_n + ((1 - \delta_n) I - \alpha_n A)y_n], \] (1.7)

where \( W_n \) is defined by (1.4), \( f \) is a contraction on \( H \), \( A \) is a strongly positive linear bounded self-adjoint operator with the coefficient \( \tilde{\gamma} > 0 \) and \( 0 < \gamma < \frac{\tilde{\gamma}}{\alpha} \). Then by using this iteration we prove the existence of a common fixed point for infinite family of nonexpansive mappings and the solution of a certain variational inequality. We need the following lemmas for the proof of our main results.

**Lemma 1.1.** The following inequality holds in a Hilbert space \( H \),

\[ \|x + y\|^2 \leq \|x\|^2 + 2(y, x + y), \forall x, y \in H. \]

**Lemma 1.2** (\cite{1}). Assume \( \{\alpha_n\} \) is a sequence of nonnegative real numbers such that \( \alpha_{n+1} \leq (1 - \gamma_n) \alpha_n + \delta_n \), \( n \geq 1 \), where \( \{\alpha_n\} \) is a sequence in \((0,1)\) and \( \delta_n \) is a sequence in \( \mathbb{R} \) such that:

1. \( \sum_{n=1}^{\infty} \frac{\alpha_n}{\gamma_n} = \infty \),
2. \( \limsup_{n \to \infty} \frac{\delta_n}{\alpha_n} \leq 0 \) or \( \sum_{n=1}^{\infty} |\delta_n| < \infty \),

then \( \lim_{n \to \infty} \alpha_n = 0 \).

**Lemma 1.3** (\cite{5}). Assume that \( A \) is a strongly positive linear bounded self-adjoint operator on a Hilbert space \( H \) with coefficient \( \tilde{\gamma} \) and \( 0 < \rho \leq \|A\|^{-1} \), then \( \|I - \rho A\| \leq 1 - \rho \tilde{\gamma} \).

**Lemma 1.4** (\cite{6}). Let \( C \) be nonempty closed convex subset of a Hilbert space, let \( T_i : C \to C \) be an infinite family of nonexpansive mappings with \( \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset \) and let \( \gamma_i \) be a real sequence such that \( 0 < \gamma_i \leq \tilde{\gamma} < 1 \) for all \( i \geq 1 \) then,

1. \( W_n \) is nonexpansive and \( F(W_n) = \bigcap_{i=1}^{n} F(T_i) \) for each \( n \geq 1 \).
2. For each \( x \in C \) and for each positive integer \( k \), the \( \lim_{n \to \infty} U_{n,k} x \) exists.
3. The mapping \( W : C \to C \) defined by,

\[ Wx := \lim_{n \to \infty} W_n x = \lim_{n \to \infty} U_{n,1} x \quad x \in C, \]

is a nonexpansive mapping satisfying \( F(W) = \bigcap_{i=1}^{\infty} F(T_i) \) and is called the \( W \)-mapping generated by \( T_1, T_2, \ldots \) and \( \gamma_1, \gamma_2, \ldots \).

**Lemma 1.5** (\cite{6}). Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \), let \( T_i : C \to C \) be an infinite family of nonexpansive mappings with \( \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset \) and let \( \gamma_i \) be a real sequence such that \( 0 < \gamma_i \leq \tilde{\gamma} < 1 \) for all \( i \geq 1 \), if \( K \) is any bounded subset of \( C \) then,

\[ \limsup_{n \to \infty} \|W x - W_n x\| = 0 \quad x \in K. \]
Lemma 1.6 ([5]). Let \( H \) be a Hilbert space, let \( A \) be a strongly positive linear bounded self-adjoint operator with coefficient \( \gamma > 0 \). Assume that \( 0 < \gamma < \frac{\gamma}{\alpha} \), let \( T \) be a nonexpansive mapping with a fixed point \( x_t \) of the contraction,
\[
x_t \leftarrow t\gamma f(x) + (I - tA)Tx.
\]
Then \( x_t \) converges strongly as \( t \to 0 \) to a fixed point \( \bar{x} \) of \( T \) which solves the variational inequality
\[
\langle (A - \gamma f)x - z, \bar{x} - z \rangle \leq 0 \quad \forall z \in F(T).
\]

Lemma 1.7 ([3]). Let \( C \) be a nonempty closed convex subset of strictly convex Banach space. Let \( \{T_i\}_{i=1}^N \) be a finite family of nonexpansive mappings of \( C \) into itself with \( \bigcap_{i=1}^N F(T_i) \neq \emptyset \), and let \( \lambda_1, \ldots, \lambda_N \) be real numbers such that \( 0 < \lambda_i < 1 \) for every \( i = 1, \ldots, N \) and \( 0 < \lambda_N \leq 1 \). Let \( K \) be the \( K \)-mapping of \( C \) into itself generated by \( T_1, \ldots, T_N \) and \( \lambda_1, \ldots, \lambda_N \). Then,
\[
F(K) = \bigcap_{i=1}^N F(T_i).
\]  

Lemma 1.8 ([7]). Let \( C \) be a nonempty closed convex subset of a Banach space. Let \( \{T_i\}_{i=1}^N \) be a finite family of nonexpansive mappings of \( C \) into itself and \( \{\lambda_{n,i}\}_{i=1}^N \) sequences in \( [0, 1] \) such that \( \lambda_{n,i} \to \lambda_i \), as \( n \to \infty \), \( (i = 1, 2, \ldots, N) \). Moreover, for every \( n \in \mathbb{N} \), \( K \) and \( K_n \) be the \( K \)-mapping generated by \( T_1, \ldots, T_N \) and \( \lambda_1, \ldots, \lambda_N \) and \( T_1, \ldots, T_N \) and \( \lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N} \), respectively. Then, for every bounded sequence \( x_n \in C \), we have \( \lim_{n \to \infty} \|K_n x_n - K x_n\| = 0 \).

2. Main Results

In this section, we prove strong convergence of the sequences \( \{x_n\} \) defined by the iteration scheme (1.7), for finding a common fixed point of infinite family of nonexpansive mappings which solves the variational inequality.

**Theorem 2.1.** Let \( C \) be a closed convex subset of a real Hilbert space \( H \). Let \( f \) be a contraction of \( C \) into itself, let \( A \) be a strongly positive linear bounded operator with coefficient \( \gamma > 0 \) and \( \{T_i : C \to C\} \) be an infinite family of nonexpansive mappings. Assume that \( 0 < \gamma < \frac{\gamma}{\alpha} \) and \( F = \bigcap_{i=1}^\infty F(T_i) \neq \emptyset \). Let \( x_0 \in C \), given that \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) and \( \{\delta_n\} \) be sequences in \( [0, 1] \) satisfying the following conditions:

\[ (C_1) \quad \lim_{n \to \infty} \alpha_n = 0 \quad \sum_{n=1}^\infty \alpha_n = \infty, \]
\[ (C_2) \quad 0 < \lim \inf_{n \to \infty} \delta_n \leq \lim \sup_{n \to \infty} \delta_n < 1, \]
\[ (C_3) \quad \sum_{n=1}^\infty |\gamma_n - \gamma_{n-1}| < \infty, \]
\[ (C_4) \quad \sum_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty, \]
\[ (C_5) \quad \sum_{n=1}^\infty |\beta_n - \beta_{n-1}| < \infty, \]
\[ (C_6) \quad (1 + \beta_n)\gamma_n - 2\beta_n > d \quad \text{for some} \quad d \in (0, 1), \]

then the sequence \( \{x_n\} \) defined by (1.7) converges strongly to \( q \in F \) which solves the variational inequality
\[
\langle f(q) - Aq, p - q \rangle \leq 0, \quad \forall p \in F.
\]

**Proof.** Since \( \alpha_n \to 0 \) as \( n \to \infty \) without loss of generality we have \( \alpha_n < (1 - \delta_n)\|A\|^{-1} \quad \forall n \geq 0 \), noticing that \( A \) is a bounded linear self-adjoint operator with,
\[
\|A\| = \sup\{|\langle Ax, x \rangle| : x \in H, \|x\| = 1\},
\]

\[ (1.8) \]
we have,
\[ < (1 - \delta_n)I - \alpha_n A)x, x > = (1 - \delta_n) < x, x > - \alpha_n < A x, x > \geq (1 - \delta_n) - \alpha_n \| A \| \geq 0, \]
then \((1 - \delta_n)I - \alpha_n A\) is positive. Also,
\[
\| (1 - \delta_n)I - \alpha_n A \| = \sup \{ < (1 - \delta_n)I - \alpha_n A)x, x > | x \in H, \| x \| = 1 \}
= \sup \{ 1 - \delta_n - \alpha_n < A x, x >, x \in H, \| x \| = 1 \}
\leq 1 - \delta_n - \alpha_n \gamma. \tag{2.1}
\]

Next we prove that \( \{ x_n \} \) is bounded. We pick \( p \in F = \bigcap_{i=1}^{\infty} F(T_i) = F(W) = F(W_n), \)
\[
\| z_n - p \| = \| \gamma_n x_n + (1 - \gamma_n) W_n x_n - p \|
= \| \gamma_n (x_n - p) + (1 - \gamma_n)(W_n x_n - W_n p) \|
\leq \gamma_n \| (x_n - p) \| + (1 - \gamma_n) \| (x_n - p) \|
= \| x_n - p \|,
\]
and we have,
\[
\| y_n - p \| = \| \beta_n x_n + (1 - \beta_n) W_n z_n - p \|
= \| \beta_n (x_n - p) + (1 - \beta_n)(W_n z_n - W_n p) \|
\leq \beta_n \| x_n - p \| + (1 - \beta_n) \| z_n - p \|
\leq \beta_n \| x_n - p \| + (1 - \beta_n) \| x_n - p \|
= \| x_n - p \|.
\]

It follows that,
\[
\| x_{n+1} - p \| = \| P_C[ \alpha_n \gamma f(x_n) + \delta_n x_n + ((1 - \delta_n)I - \alpha A)y_n] - P_C(p) \|
\leq \| \alpha_n \gamma f(x_n) + \delta_n x_n + ((1 - \delta_n)I - \alpha A)y_n - p \|
= \| \alpha_n (\gamma f(x_n - Ap) + \delta_n (x_n - p) + ((1 - \delta_n)I - \alpha A)(y_n - p)) \|
\]
by (2.1) we have,
\[
\leq \alpha_n \| \gamma f(x_n - Ap) \| + \delta_n \| x_n - p \| + (1 - \delta_n - \alpha_n \gamma) \| y_n - p \|
\leq \alpha_n \gamma \| f(x_n) - f(p) \| + \alpha_n \| \gamma f(p - Ap) \| + \delta_n \| x_n - p \|
+ (1 - \delta_n - \alpha_n \gamma) \| x_n - p \|
\leq \alpha_n \gamma \| x_n - p \| + \alpha_n \| \gamma f(p - Ap) \| + (1 - \alpha_n \gamma) \| x_n - p \|
= [1 - \alpha_n (\gamma - \gamma \alpha)] \| x_n - p \| + \alpha_n \| \gamma f(p - Ap) \| .
\]

By simple induction we have \( \| x_n - p \| \leq \max \{ \| x_0 - p \|, \frac{\| Ap - f(p) \|}{\gamma - \gamma \alpha} \}, \) which gives that the sequence \( \{ x_n \} \) is bounded so are \( \{ y_n \} \) and \( \{ z_n \}. \) Next we claim that, \( \lim_{n \to \infty} \| x_{n+1} - x_n \| = 0. \) We know that,
\[
z_n = \gamma_n x_n + (1 - \gamma_n) W_n x_n ,
z_{n-1} = \gamma_{n-1} x_{n-1} + (1 - \gamma_{n-1}) W_{n-1} x_{n-1}.
\]
So we obtain,
\[
z_n - z_{n-1} = (1 - \gamma_n)(W_n x_n - W_{n-1} x_{n-1}) + \gamma_n (x_n - x_{n-1})
+ (\gamma_n - \gamma_{n-1})(x_{n-1} - W_{n-1} x_{n-1}).
\]
This implies that,

\[
\|z_n - z_{n-1}\| \leq (1 - \gamma_n)\|W_n x_n - W_{n-1} x_{n-1}\| + \gamma_n \|x_n - x_{n-1}\|
\]  
\[
+ |\gamma_n - \gamma_{n-1}|\|x_{n-1} - W_{n-1} x_{n-1}\|
\]  
\[
= (1 - \gamma_n)\|W_n x_n - W_n x_{n-1} + W_n x_{n-1} - W_{n-1} x_{n-1}\|
\]  
\[
+ \gamma_n \|x_n - x_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|x_{n-1} - W_{n-1} x_{n-1}\|
\]  
\[
\leq (1 - \gamma_n)\|W_n x_n - W_{n-1} x_{n-1}\|
\]  
\[
+ (1 - \gamma_n)\|W_n x_{n-1} - W_{n-1} x_{n-1}\|
\]  
\[
+ \gamma_n \|x_n - x_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|x_{n-1} - W_{n-1} x_{n-1}\|.
\]

On the other hand we have,

\[
\|W_n x_{n-1} - W_{n-1} x_{n-1}\| = \|\gamma_1 T_1 U_n,2 x_{n-1} - \gamma_1 T_1 U_{n-1,2} x_{n-1}\|
\]  
\[
\leq \gamma_1 \|U_n,2 x_{n-1} - U_{n-1,2} x_{n-1}\|
\]  
\[
= \gamma_1 \|\gamma_2 T_2 U_n,3 x_{n-1} - \gamma_2 T_2 U_{n-1,3} x_{n-1}\|
\]  
\[
\leq \gamma_1 \gamma_2 \|U_n,3 x_{n-1} - U_{n-1,3} x_{n-1}\|
\]  
\[
\vdots
\]  
\[
\leq \gamma_1 \gamma_2 \cdots \gamma_n \|U_n,n x_{n-1} - U_{n-1,n} x_{n-1}\|
\]  
\[
\leq M_1 \prod_{i=1}^{n-1} \gamma_i,
\]

where \(M_1 \geq 0\) is an appropriate constant such that,

\[
\|U_{n,n} x_{n-1} - U_{n-1,n} x_{n-1}\| \leq M_1 \ \forall \ n \geq 0.
\]

Note that the boundedness of \(x_n\) and the nonexpansivity of \(T_n\) ensure the existence of \(M_1\). So we have,

\[
\|z_n - z_{n-1}\| \leq \gamma_n \|x_n - x_{n-1}\| + (1 - \gamma_n) M_1 \prod_{i=1}^{n-1} \gamma_i
\]  
\[
+ (1 - \gamma_n)\|x_n - x_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|x_{n-1} - W_{n-1} x_{n-1}\|
\]  
\[
= \|x_n - x_{n-1}\|
\]  
\[
+ (1 - \gamma_n) M_1 \prod_{i=1}^{n-1} \gamma_i + |\gamma_n - \gamma_{n-1}|\|x_{n-1} - W_{n-1} x_{n-1}\|.
\]

Similar to (2.2), we have,

\[
\|U_{n,n} z_{n-1} - U_{n-1,n} z_{n-1}\| \leq M_2.
\]

So,

\[
\|y_n - y_{n-1}\| = \|\beta_n x_n + (1 - \beta_n) W_n z_n - \beta_{n-1} x_{n-1} - (1 - \beta_{n-1}) W_{n-1} z_{n-1}\|
\]  
\[
= \|\beta_n x_n - \beta_n x_{n-1} + \beta_n x_{n-1} - \beta_{n-1} x_{n-1}\|
\]  
\[
+ (1 - \beta_n)(W_n z_n - W_{n-1} z_{n-1})
\]  
\[
+ (1 - \beta_n)(W_n z_{n-1} - W_{n-1} z_{n-1})
\]  
\[
+ (1 - \beta_{n-1})W_{n-1} z_{n-1} - (1 - \beta_{n-1}) W_{n-1} z_{n-1}\|
\]  
\[
\leq \|\beta_n (x_n - x_{n-1}) + (\beta_n - \beta_{n-1}) x_{n-1}\|
\]  
\[
+ (1 - \beta_n)(W_n z_n - W_{n-1} z_{n-1})
\]
\[
\|x_{n+1} - x_n\| = \|P_C[\alpha_n \gamma f(x_n) + \delta_n x_n + ((1 - \delta_n)I - \alpha_n A)y_n] \\
- P_C[\alpha_n \gamma f(x_n)] \\
+ \delta_n x_n + ((1 - \delta_n)I - \alpha_n A)y_n\| \\
\leq \|\alpha_n \gamma f(x_n) + \delta_n x_n + ((1 - \delta_n)I - \alpha_n A)y_n\| \\
- \alpha_n \gamma f(x_n) - \delta_n x_n - ((1 - \delta_n)I - \alpha_n A)y_n\| \\
\leq \|(1 - \delta_n)I - \alpha_n A)(y_n - y_{n-1}) \\
- ((\delta_n - \delta_n) y_{n-1} + (\alpha_n - \alpha_n) A y_{n-1}) \\
+ \gamma \alpha_n f(x_n) - f(x_{n-1}) + \gamma (\alpha_n - \alpha_n)f(x_{n-1}) \\
+ \delta_n x_n - \delta_n x_n + \delta_n x_n - \delta_n x_{n-1}\| \\
\leq (1 - \delta_n - \alpha_n \gamma)\|y_n - y_{n-1}\| + |\delta_n - \delta_n|\|y_{n-1}\| \\
+ |\alpha_n - \alpha_n|\|A y_{n-1}\| + \gamma |\alpha_n|\|x_n - x_{n-1}\| \\
+ \gamma |\alpha_n - \alpha_n|\|f(x_{n-1})\| + \delta_n \|x_n - x_{n-1}\| + |\delta_n - \delta_n|\|x_{n-1}\| \\
\leq (1 - \delta_n - \alpha_n \gamma)\|x_n - x_{n-1}\| + (1 - \delta_n)\|x_n - x_{n-1}\| - W_{n-1} x_{n-1}\| \\
+ (1 - \beta_n)(1 - \gamma_n)M_1 \prod_{i=1}^{n-1} \gamma_i + (1 - \beta_n)M_2 \prod_{i=1}^{n-1} \gamma_i \\
+ |\beta_n - \beta_n|\|x_n - x_{n-1}\| - W_{n-1} x_{n-1}\| \\
+ |\alpha_n - \alpha_n|\|A y_{n-1}\| \\
+ \gamma |\alpha_n - \alpha_n|\|f(x_{n-1})\| + \delta_n \|x_n - x_{n-1}\| + |\delta_n - \delta_n|\|x_{n-1}\| \\
= (1 - \alpha_n \gamma)\|x_n - x_{n-1}\| \\
+ (1 - \delta_n - \alpha_n \gamma)(1 - \beta_n)\|x_n - x_{n-1}\| - W_{n-1} x_{n-1}\| \\
+ (1 - \beta_n)(1 - \gamma_n)M_1 \prod_{i=1}^{n-1} \gamma_i + (1 - \beta_n)M_2 \prod_{i=1}^{n-1} \gamma_i \\
\]
so \lim sup\sup_n x_n = 0.

Therefore, we show that \lim sup \sup_n \|x_n - x_n - W_n z_n\| = 0.

Now by Lemma 1.2 and C_3, C_4, C_5 we have \|x_n - x_n - W_n z_n\| \to 0. On the other hand,

\begin{align*}
\|x_{n+1} - y_n\| &= \|P_C[\alpha_n \gamma f(x_n) + \delta_n x_n + ((1 - \delta_n)I - \alpha_n A)y_n] - P_C(y_n)\| \\
&\leq \|\alpha_n \gamma f(x_n) + \delta_n x_n + ((1 - \delta_n)I - \alpha_n A)y_n - y_n\| \\
&= \|\alpha_n \gamma f(x_n) + \delta_n x_n - \delta_n x_{n+1} + \delta_n x_{n+1} + y_n - \delta_n y_n - y_n - \alpha_n A y_n\| \\
&\leq \|\alpha_n \gamma f(x_n) + \delta_n x_n - \delta_n x_{n+1} + \delta_n x_{n+1} - y_n - \alpha_n A y_n\|. 
\end{align*}

So, \|x_{n+1} - y_n\| \leq \alpha_n \|\gamma f(x_n) - A y_n\| + \frac{\delta_n}{(1 - \delta_n)} \|x_n - x_{n+1}\|, which implies, \|x_{n+1} - y_n\| \to 0. Also we have \|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|, which implies \|x_n - y_n\| \to 0. Notice that,

\begin{align*}
\|z_n - x_n\| &= \|\gamma_n x_n + (1 - \gamma_n) W_n x_n\| = \|\gamma_n x_n + (1 - \gamma_n) W_n x_n\| \\
&\leq \|x_n - y_n\| + \|y_n - W_n z_n\| = \|x_n - y_n\| + \|y_n - W_n z_n\|. 
\end{align*}

By two above equalities we have,

\begin{align*}
\|W_n x_n - x_n\| &\leq \|x_n - y_n\| + \|y_n - W_n x_n\| \\
&\leq \|x_n - y_n\| + \|y_n - W_n z_n\| + \|W_n z_n - W_n x_n\| \\
&\leq \|x_n - y_n\| + \beta_n \|x_n - W_n x_n\| + \beta_n \|W_n x_n - W_n z_n\| \\
&\quad + \|z_n - x_n\| \\
&\leq \|x_n - y_n\| + \beta_n \|x_n - W_n x_n\| + (1 + \beta_n) \|z_n - x_n\| \\
&\leq \|x_n - y_n\| + \beta_n \|x_n - W_n x_n\| + (1 - \gamma_n)(1 + \beta_n) \|W_n x_n - x_n\|.
\end{align*}

Therefore,

\begin{align*}
[(1 + \beta_n) \gamma_n - 2 \beta_n] \|W_n x_n - x_n\| &\leq \|x_n - y_n\| \to 0,
\end{align*}

so \lim_{n \to \infty} \|W_n x_n - x_n\| = 0.

Furthermore we have,

\begin{align*}
\|W x_n - x_n\| &\leq \|W x_n - W_n x_n\| + \|W_n x_n - x_n\|, 
\end{align*}

hence \lim_{n \to \infty} \|W x_n - x_n\| = 0.

We show that \lim sup_{n \to \infty} \langle \gamma f(q) - Aq, x_n - q \rangle \leq 0, where q = \lim_{t \to 0} x_t and x_t is the fixed point of the
contractible \( x \mapsto t \gamma f(x) + (I - tA)Wx \). We have, \( \|x_t - x_n\| = \|(I - tA)(Wx_t - x_n) + t(\gamma f(x_t) - Ax_n)\| \) and by Lemma 1.1,

\[
\|x_t - x_n\|^2 = \|(I - tA)(Wx_t - x_n) + t(\gamma f(x_t) - Ax_n)\|^2 \\
\leq (1 - t\hat{\gamma})^2\|Wx_t - x_n\|^2 + 2t(\gamma f(x_t) - Ax_n, x_t - x_n) \\
\leq (1 - 2\gamma t + (\gamma t)^2)\|x_t - x_n\|^2 + f_n(t) + 2t\langle \gamma f(x_t) - Ax_t, x_t - x_n \rangle \\
+ 2t\langle Ax_t - Ax_n, x_t - x_n \rangle
\]  

(2.3)

where \( f_n(t) = (2\|x_t - x_n\| + \|x_n - Wx_n\|)\|x_n - Wx_n\| \to 0 \) (as \( n \to \infty \)). Since \( A \) is strongly positive linear mapping, so we have,

\[
\langle Ax_t - Ax_n, x_t - x_n \rangle = \langle A(x_t - x_n), x_t - x_n \rangle \geq \hat{\gamma}\|x_t - x_n\|^2.
\]

From (2.3) we have,

\[
2\langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leq (\gamma^2 t^2 - 2\gamma t)\|x_t - x_n\|^2 + f_n(t) \\
+ 2t\langle Ax_t - Ax_n, x_t - x_n \rangle \\
\leq (\gamma t^2)\langle A(x_t - x_n), x_t - x_n \rangle + f_n(t) \\
+ 2t\langle A(x_t - x_n), x_t - x_n \rangle \\
= \gamma t^2\langle A(x_t - x_n), x_t - x_n \rangle + f_n(t),
\]

which implies, \( \langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leq \frac{\gamma t}{2}\langle A(x_t) - A(x_n), x_t - x_n \rangle + \frac{f_n(t)}{2t} \).

Letting \( n \to \infty \),

\[
\limsup_{n \to \infty}\langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leq \frac{t}{2} M_3,
\]

(2.4)

where \( M_3 \) is a constant such that, \( \hat{\gamma}\langle Ax_t - Ax_n, x_t - x_n \rangle \leq M_3, \forall t \in (0, \min\{|A|^{-1}, 1\}) \) and \( n \geq 1 \), taking \( t \to 0 \), from (2.4) we have,

\[
\limsup_{t \to 0}\limsup_{n \to \infty}\langle Ax_t - \gamma f(x_t), x_t - x_n \rangle \leq 0.
\]

(2.5)

On the other hand we have,

\[
\langle \gamma f(q) - Aq, x_n - q \rangle = \langle \gamma f(q) - Aq, x_n - q \rangle \\
- \langle \gamma f(q) - Aq, x_n - x_t \rangle + \langle \gamma f(q) - Aq, x_n - x_t \rangle \\
- \langle \gamma f(q) - Ax_t, x_n - x_t \rangle + \langle \gamma f(q) - Ax_t, x_n - x_t \rangle \\
- \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle.
\]

So,\n
\[
\langle \gamma f(q) - Aq, x_n - q \rangle = \langle \gamma f(q) - Aq, x_t - q \rangle + \langle Ax_t - Aq, x_n - x_t \rangle + \langle \gamma f(q) - \gamma f(x_t), x_n - x_t \rangle + \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle.
\]

Hence,

\[
\limsup_{n \to \infty} \langle \gamma f(q) - Aq, x_n - q \rangle \leq \|\gamma f(q) - Aq\|\|x_t - q\| + \|A\|\|x_t - q\| \limsup_{n \to \infty} \|x_n - x_t\| \\
+ \alpha\|q - x_t\| \limsup_{n \to \infty} \|x_n - x_t\| + \limsup_{n \to \infty} \|\gamma f(x_t) - Ax_t, x_n - x_t\|.
\]

Therefore from (2.5) we have,
\[
\limsup_{n \to \infty} \langle \gamma f(q) - Aq, x_n - q \rangle = \limsup_{t \to 0} \limsup_{n \to \infty} \langle \gamma f(q) - Aq, x_n - q \rangle \\
\leq \limsup_{t \to 0} \| \gamma f(q) - Aq \| \| x_t - q \| \\
+ \limsup_{t \to 0} \| A \| \| x_t - q \| \limsup_{n \to \infty} \| x_n - x_t \| \\
+ \limsup_{t \to 0} \gamma \alpha \| q - x_t \| \limsup_{n \to \infty} \| x_n - x_t \| \\
+ \limsup_{t \to 0} \limsup_{n \to \infty} \langle \gamma f(x_t) - Ax_t, x_n - x_t \rangle \leq 0.
\]

Similarly,
\[
\langle \gamma f(q) - Aq, y_n - q \rangle = \langle \gamma f(q) - Aq, y_n - x_n \rangle + \langle \gamma f(q) - Aq, x_n - q \rangle \\
\leq \| \gamma f(q) - Aq \| \| y_n - x_n \| + \langle \gamma f(q) - Aq, x_n - q \rangle,
\]
then, \( \limsup_{n \to \infty} \langle \gamma f(q) - Aq, y_n - q \rangle \leq 0. \) Finally we prove that \( x_n \to q. \)
\[
\| x_{n+1} - q \|^2 = \| P_C [\alpha_n \gamma f(x_n) + \delta_n x_n + ((1 - \delta_n) I - \alpha_n A) y_n] - P_C(q) \|^2 \\
\leq \| \alpha_n \gamma f(x_n) + \delta_n x_n + ((1 - \delta_n) I - \alpha_n A) y_n - q \|^2 \\
= \| \alpha_n \gamma f(x_n - Aq) + \delta_n (x_n - q) + ((1 - \delta_n) I - \alpha_n A) (y_n - q) \|^2 \\
= \| ((1 - \delta_n) I - \alpha_n A) (y_n - q) + \delta_n (x_n - q) + \alpha_n \gamma f(x_n - Aq) \|^2 \\
= \| ((1 - \delta_n) I - \alpha_n A) (y_n - q) + \delta_n (x_n - q) \|^2 \\
+ \alpha_n^2 \| \gamma f(x_n - Aq) \|^2 + 2 \delta_n \alpha_n \langle x_n - q, \gamma f(x_n - Aq) \rangle \\
+ 2 \alpha_n \langle ((1 - \delta_n) I - \alpha_n A) (y_n - q), \gamma f(x_n - Aq) \rangle \\
\leq \| (1 - \delta_n) - \alpha_n \gamma \| y_n - q \| + \delta_n \| x_n - q \|^2 \\
+ \alpha_n^2 \| \gamma f(x_n - Aq) \|^2 + 2 \delta_n \alpha_n \langle x_n - q, \gamma f(x_n - Aq) \rangle \\
+ 2 \alpha_n \langle ((1 - \delta_n) I - \alpha_n A) (y_n - q), \gamma f(x_n - Aq) \rangle \\
= \| (1 - \delta_n) - \alpha_n \gamma \| x_n - q \| + \delta_n \| x_n - q \|^2 \\
+ \alpha_n^2 \| \gamma f(x_n - Aq) \|^2 + 2 \delta_n \alpha_n \gamma \| x_n - q \|^2 \\
+ 2 \delta_n \alpha_n \langle x_n - q, \gamma f(q) - Aq \rangle + 2 (1 - \delta_n) \gamma \alpha_n \| x_n - q \|^2 \\
+ 2 (1 - \delta_n) \alpha_n \langle y_n - q, \gamma f(q) - Aq \rangle - 2 \alpha_n^2 \langle A (y_n - q), \gamma f(q) - Aq \rangle \\
= \| (1 - \alpha_n \gamma) \|^2 + 2 \delta_n \alpha_n \gamma \alpha + 2 (1 - \delta_n) \gamma \alpha_n \| x_n - q \|^2 \\
+ \alpha_n^2 \| \gamma f(x_n - Aq) \|^2 + 2 \delta_n \alpha_n \langle x_n - q, \gamma f(q) - Aq \rangle \\
+ 2 (1 - \delta_n) \alpha_n \langle y_n - q, \gamma f(q) - Aq \rangle - 2 \alpha_n^2 \langle A (y_n - q), \gamma f(q) - Aq \rangle \\
\leq \| 1 - 2 (\bar{\gamma} - \alpha \gamma) \alpha_n \| x_n - q \|^2 + \bar{\gamma}^2 \alpha_n^2 \| x_n - q \|^2 \\
+ \alpha_n^2 \| \gamma f(x_n - Aq) \|^2 + 2 \delta_n \alpha_n \langle x_n - q, \gamma f(q) - Aq \rangle \\
+ 2 (1 - \delta_n) \alpha_n \langle y_n - q, \gamma f(q) - Aq \rangle + 2 \alpha_n \langle A (y_n - q), \gamma f(q) - Aq \rangle \\
\leq \| 1 - 2 (\bar{\gamma} - \alpha \gamma) \alpha_n \| x_n - q \|^2 + \bar{\gamma}^2 \alpha_n^2 \| x_n - q \|^2 \\
+ \| \gamma f(x_n - Aq) \|^2 + 2 \| A (y_n - q) \| \| \gamma f(q) - Aq \| + 2 \delta_n \langle x_n - q, \gamma f(q) - Aq \rangle \\
+ 2 (1 - \delta_n) \langle y_n - q, \gamma f(q) - Aq \rangle \}.
Since \( \{x_n\}, \{f(x_n)\} \) and \( \|y_n - p\| \) are bounded, we can take a constant \( M_4 > 0 \) such that,
\[
\tilde{\gamma}^2 \|x_n - q\|^2 + \|\gamma f(x_n) - A^*q\|^2 + 2 \|A(y_n - q)\| \|\gamma f(q) - Aq\| \leq M_4, \quad \forall \ n \geq 0,
\]
then it follows that, \( \|x_{n+1} - q\|^2 \leq [1 - (2(\tilde{\gamma} - \gamma)\alpha_\nu)\|x_n - q\|^2 + \alpha_\nu \sigma_\nu \), where,
\[
\sigma_\nu = 2\delta_\nu \langle x_n - q, \gamma f(q) - Aq \rangle + 2(1 - \delta_\nu) \langle y_n - q, \gamma f(q) - Aq \rangle + \alpha_\nu M_4.
\]
Finally, we have \( \limsup_{n \to \infty} \sigma_\nu \leq 0 \) and by Lemma 1.2 \( x_n \to q \).

Similar proof shows that the followings composite iteration converges to \( q \in F \), which solves variational inequality,
\[
\begin{align*}
x_0 = x & \in C \text{ chosen arbitrary,} \\
z_n = & \lambda_n x_n + (1 - \lambda_n)K_n x_n, \\
y_n = & \beta_n x_n + (1 - \beta_n)K_n z_n, \\
x_{n+1} = & \text{Prox}[\alpha_\nu \gamma f(x_n) + \delta_\nu x_n + (1 - \delta_\nu)I - \alpha_\nu A]y_n.
\end{align*}
\]

**Corollary 2.2.** Let \( C \) be a closed convex subset of a real Hilbert space \( H \). Let \( f \) be a contraction of \( C \) into itself, let \( A \) be a strongly positive linear bounded operator with coefficient \( \tilde{\gamma} > 0 \) and \( \{T_i : C \to C\} \) be a finite family of nonexpansive mappings. Assume that \( 0 < \gamma < \frac{\tilde{\gamma}}{2} \) and \( F = \bigcap_{i=1}^N F(T_i) \neq \emptyset \). Let \( x_0 \in C \), given that \( \{\alpha_\nu\}, \{\beta_\nu\} \) and \( \{\delta_\nu\} \) be sequences in \([0,1]\) satisfying the following conditions:
\[
\begin{align*}
& (C_1) \quad \lim_{n \to \infty} \alpha_\nu = 0, \sum_{n=1}^\infty \alpha_\nu = \infty, \\
& (C_2) \quad 0 < \liminf_{n \to \infty} \delta_\nu \leq \limsup_{n \to \infty} \delta_\nu < 1, \\
& (C_3) \quad \sum_{n=1}^\infty |\lambda_{n,i} - \lambda_{n-1,i}| < \infty, \text{ for all } i = 1, 2, \ldots, N, \\
& (C_4) \quad \sum_{n=1}^\infty |\alpha_\nu - \alpha_{\nu-1}| < \infty, \\
& (C_5) \quad \sum_{n=1}^\infty |\beta_\nu - \beta_{\nu-1}| < \infty,
\end{align*}
\]

If \( \{x_n\}_{n=1}^\infty \) is the composite process defined by (2.6), then the sequence \( \{x_n\}_{n=1}^\infty \) converges strongly to \( q \in F \), which solves variational inequality \( \langle \gamma f(q) - Aq, p - q \rangle \leq 0, \forall p \in F \).

If \( \lambda_n = 1 \) and \( \delta_\nu = 0 \) in Corollary 2.2, then we get the result of Singhthong and Suantai [7].

**Corollary 2.3.** Let \( H \) be a Hilbert space, \( C \) a closed convex subset of \( H \). Let \( A \) be a strongly positive linear bounded operator with coefficient \( \tilde{\gamma} \geq 0 \), and \( f \) is a contraction. Let \( \{T_i\}_{i=1}^N \) be a finite family of nonexpansive mappings of \( C \) into itself and let \( K_n \) be defined by (1.6). Assume that \( 0 < \gamma < \frac{\tilde{\gamma}}{2} \) and \( F = \bigcap_{i=1}^N F(T_i) \neq \emptyset \). Let \( x_1 \in C \), given that \( \{\alpha_\nu\}_{\nu=0}^\infty \) and \( \{\beta_\nu\}_{\nu=0}^\infty \) are sequences in \((0,1)\), and suppose that the following conditions are satisfied:
\[
\begin{align*}
& (C_1) \quad \alpha_\nu \to 0, \sum_{\nu=1}^\infty \alpha_\nu = \infty, \\
& (C_2) \quad 0 < \liminf_{\nu \to \infty} \beta_\nu \leq \limsup_{\nu \to \infty} \beta_\nu < 1, \\
& (C_3) \quad \sum_{\nu=1}^\infty |\gamma_{n,i} - \gamma_{n-1,i}| < \infty \text{ for all } i = 1, 2, \ldots, N, \\
& (C_4) \quad \sum_{\nu=1}^\infty |\alpha_{\nu+1} - \alpha_\nu| < \infty, \\
& (C_5) \quad \sum_{\nu=1}^\infty |\beta_{\nu+1} - \beta_\nu| < \infty.
\end{align*}
\]

If \( \{x_n\}_{n=1}^\infty \) is the composite process defined by (1.5), then the sequence \( \{x_n\} \) converges strongly to \( q \in F \), which solves the variational inequality \( \langle \gamma f(q) - Aq, p - q \rangle \leq 0, \forall p \in F \).

3. Simulation examples

In this section, we give three numerical examples to support the theoretical results. The iterations have been carried out on MATLAB 7.12. Here we recall \( r(n) = \log_{10} \|x_{n+1} - x_n\| \) and \( \delta(n) = \log_{10} \|\frac{x_{n+1} - x_n}{\|x_{n+1} - x_n\|}\)
(i.e. \( \delta(n) \) is relative error), where \( x^* \) is a fixed point of \( W_n\)-mapping or \( K\)-mapping. In the following, we assume \( \gamma_1 = \frac{1}{2}, \gamma_2 = \frac{1}{3}, \gamma_3 = \frac{1}{4} \), and \( x_0 = 3 \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
 & x^* & \text{iteration} & T_1(x^*) & T_2(x^*) \\
\hline
W_n \text{ mapping} & 0.75290 & 25 & 0.6837577884 & 0.7297090424 \\
K \text{ mapping} & 0.71491 & 19 & 0.6555494556 & 0.7551522437 \\
\hline
\end{array}
\]

Table 1: \( T_1(x) = \sin(x) \) and \( T_2(x) = \cos(x) \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
 & x^* & \text{iteration} & T_1(x^*) & T_3(x^*) \\
\hline
W_n \text{ mapping} & 0.0089628 & 44834 & 0.0089626800 & 0.0089625600 \\
K \text{ mapping} & 0.0080118 & 40066 & 0.0080117142 & 0.0080116285 \\
\hline
\end{array}
\]

Table 2: \( T_1(x) = \sin(x) \) and \( T_3(x) = \tan^{-1}(x) \).

\[
\begin{array}{|c|c|c|c|c|}
\hline
 & x^* & \text{iteration} & T_1(x^*) & T_2(x^*) & T_3(x^*) \\
\hline
W_n \text{ mapping} & 0.59403 & 85 & 0.5597051868 & 0.8286918026 & 0.5360182305 \\
K \text{ mapping} & 0.67735 & 18 & 0.6267302508 & 0.7792362880 & 0.5953623347 \\
\hline
\end{array}
\]

Table 3: \( T_1(x) = \sin(x) \), \( T_2(x) = \cos(x) \) and \( T_3(x) = \tan^{-1}(x) \).

Figure 1: The results obtained for \( T_1 \) and \( T_2 \).

Figure 2: The results obtained for \( T_1 \) and \( T_3 \).
4. Conclusion

Finding the fixed point of nonexpansive mappings and variational inequalities is so important in many fields. In this paper, we have constructed an iterative algorithm for finding a common fixed point of an infinite family of nonexpansive mappings and a solution of certain variational inequality. Finally, some numerical examples were presented to support the theoretical results of this paper. Moreover, these examples compare the error and speed of convergence of $W_n$-mapping and $K$-mapping.

References