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On the stability of the cubic functional equation on n-Abelian groups

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Abstract

Most the literature on the stability of the cubic functional equation focus on the case where the relevant domain is a normed space. In this paper, we investigate the stability of the cubic functional equation on n-Abelian groups.

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1. Introduction

In 1940, S. M. Ulam [19] proposed the following question concerning the stability of group homomorphisms:
Let G_1 be a group and (G_2, d) a metric group. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ such that $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?
In the next year D. H. Hyers [13] answers the problem of Ulam under the assumption that the groups are Banach spaces:
Let X be a normed space and Y a Banach space. Suppose that for some $\varepsilon > 0$, the mapping $f : X \to Y$ satisfies $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in X$. Then there exists a unique additive mapping $T : X \to Y$ such that $\|f(x) - T(x)\| \leq \varepsilon$ for all $x \in X$.
In 1978, Th. M. Rassias [17] formulated and proved the following theorem:

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Let X and Y be real normed spaces with Y complete, let $f : X \to Y$ be a mapping such that, for each fixed $x \in X$, the mapping $h(t) = f(tx)$ is continuous on $\mathbb{R}$, and let $\varepsilon \geq 0$ and $p \in [0, 1)$ be such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \left(\|x\|^p + \|y\|^p\right),$$

for all $x, y \in X$, then there exists a unique linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \varepsilon \frac{\|x\|^p}{1 - 2^{p-1}},$$

for all $x \in X$.
Next Gavruta [12] proved the generalized Hyers-Ulam-Rassias theorem. He replaced $\varepsilon(\|x\|^p + \|y\|^p)$ in the theorem of Rassias by $\phi(x, y)$ where ϕ is a function such that $\sum\limits_{k=0}^{\infty} \frac{1}{2^k}\phi(2^k x, 2^k y)$ is finite for all $x, y \in X$. Jun and Kim [14] introduced the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{1.1}$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.1).
Every solution of the cubic functional equation is said to be a cubic mapping.
M. Eshaghi Gordji and M. Bavand Savadkouhi [4] proved the generalized Hyers-Ulam-Rassias stability of the cubic and quartic functional equations in non-Archimedean normed spaces.
Moreover the generalized Hyers-Ulam-Rassias stability of the mixed type cubic-quartic functional equations in non-Archimedean normed spaces was investigated in [5].
During the last decades several stability problems of functional equations have been investigated. The reader is referred to [6, 7, 15] and references therein for detailed information on stability of functional equations.
The first paper extending the Hyers result to a class of non-Abelian groups and semigroups was [8]. The notion of (ψ, γ)-stability of the Cauchy functional equation was introduced in [9]. In [9], among other results, it was proved that the Cauchy functional equation

$$f(xy) = f(x) + f(y),$$

is (ψ, γ)-stable on any Abelian group, as well as on any meta-Abelian (step-two nilpotent) group.

2. Preliminaries

In this section, we consider the stability of the cubic functional equation

$$f\left(x^2 y\right) + f\left(x^2 y^{-1}\right) - 2f(xy) - 2f\left(xy^{-1}\right) - 12f(x) = 0, \tag{2.1}$$

for the pair (G, X) where G is an arbitrary group and X is a real Banach space. Every solution of the functional equation (2.1) is said to be a cubic mapping. We prove that if G is an n-Abelian group with $n \in \mathbb{N}$, then the cubic functional equation (2.1) is stable on group G. The Jun and Kim result [14] is a particular case of this result. In this sequel we will write the arbitrary group G in multiplicative notation. Throughout the section X denotes a Banach space.

Definition 2.1. The cubic functional equation (2.1) is said to be stable for the pair (G, X), (write (G, X) is CS for short) if for every function $f : G \to X$ such that

$$\left\|f\left(x^2 y\right) + f\left(x^2 y^{-1}\right) - 2f(xy) - 2f\left(xy^{-1}\right) - 12f(x)\right\| \leq \delta, \tag{2.2}$$

for all $x, y \in G$ and some $\delta \geq 0$, there is a solution T of the functional equation (2.1) and a constant $\epsilon \geq 0$ dependent only on δ satisfying

$$\|f(x) - T(x)\| \leq \epsilon. \tag{2.3}$$

Lemma 2.2. *Let G be an Abelian group. If $f : G \to X$ satisfies the inequality (2.2) for all $x, y \in G$ and some $\delta \geq 0$, then there exists a unique cubic mapping $T : G \to X$ such that*

$$\|f(x) - T(x)\| \leq \frac{1}{14}\delta, \tag{2.4}$$

for all $x \in G$.

Proof. Put $y = e$ in (2.2) to get

$$\|f(x^2) - 8f(x)\| \leq \frac{1}{2}\delta. \tag{2.5}$$

Let $n, m \in \mathbb{N}$ with $n > m$. Then from (2.5), we have

$$\left\| \frac{1}{8^m} f\left(x^{2^m}\right) - \frac{1}{8^n} f\left(x^{2^n}\right) \right\| \leq \sum_{i=m}^{n-1} \left\| \frac{1}{8^i} f\left(x^{2^i}\right) - \frac{1}{8^{i+1}} f\left(x^{2^{i+1}}\right) \right\| \leq \frac{1}{16}\delta \sum_{i=m}^{n-1} \frac{1}{8^i}. \tag{2.6}$$

Therefore the sequence $\left(\frac{1}{8^n} f\left(x^{2^n}\right)\right)$ is Cauchy and so is convergent in the Banach space X. Set

$$T(x) := \lim_{n\to\infty} \frac{1}{8^n} f\left(x^{2^n}\right).$$

Next put $m = 0$ in (2.6) to get

$$\left\| f(x) - \frac{1}{8^n} f\left(x^{2^n}\right) \right\| \leq \frac{1}{14}\delta \left(1 - \left(\frac{1}{8}\right)^n\right).$$

Letting n tend to infinity, we obtain

$$\|f(x) - T(x)\| \leq \frac{1}{14}\delta.$$

The function T is a cubic mapping. Indeed for any $n \in \mathbb{N}$ and any $x, y \in G$, we have

$$\left\| \frac{1}{8^n} f\left(\left(x^2 y\right)^{2^n}\right) + \frac{1}{8^n} f\left(\left(x^2 y^{-1}\right)^{2^n}\right) - 2\frac{1}{8^n} f\left((xy)^{2^n}\right) - 2\frac{1}{8^n} f\left(\left(xy^{-1}\right)^{2^n}\right) - 12\frac{1}{8^n} f\left(x^{2^n}\right) \right\| \leq \frac{1}{8^n}\delta,$$

because G is an Abelian group.
Letting n tend to infinity, we obtain

$$T\left(x^2 y\right) + T\left(x^2 y^{-1}\right) - 2T(xy) - 2T\left(xy^{-1}\right) - 12T(x) = 0,$$

for all $x, y \in G$. Hence T is a cubic mapping.
To prove the uniqueness assertion, assume that there exists a mapping S satisfying (2.4). It is easy to verify that every cubic mapping g satisfies $g\left(x^k\right) = k^3 g(x)$ for any $x \in G$ and any $k \in \mathbb{N}$. So

$$\left\| T(x) - S(x) \right\| = \frac{1}{n^3} \left\| T\left(x^n\right) - S\left(x^n\right) \right\| \leq \frac{1}{n^3} \left\| T\left(x^n\right) - f\left(x^n\right) \right\| + \frac{1}{n^3} \left\| f\left(x^n\right) - S\left(x^n\right) \right\| \leq \frac{1}{n^3}\left(\frac{1}{7}\delta\right),$$

for every $x \in G$ and any $n \in \mathbb{N}$. Hence $T = S$. This proves the uniqueness assertion. □

Lemma 2.3. *Assume that $f : G \to X$ satisfies the inequality (2.2) for all $x, y \in G$ and some $\delta \geq 0$. Then the limit*

$$k(x) = \lim_{n\to\infty} \frac{1}{8^n} f\left(x^{2^n}\right), \tag{2.7}$$

exists for all $x \in G$, and

$$\|f(x) - k(x)\| \leq \frac{1}{14}\delta \quad \textit{and} \quad k(x^2) = 8k(x), \tag{2.8}$$

for all $x \in G$. The function k with conditions (2.8) is unique.

Proof. Let x be a fixed element in G. If we consider the cyclic subgroup $\langle x\rangle$ of G, then by Lemma 2.2, we conclude the existence of a mapping $k : G \to X$ such that k is cubic and (2.7) and (2.8) are satisfied. If there exists a mapping $k' : G \to X$ such that

$$\|f(x) - k'(x)\| \leq \frac{1}{14}\delta,$$

and

$$k'\left(x^2\right) = 8k'(x),$$

for all $x \in G$, then by induction we obtain

$$k\left(x^{2^n}\right) = 8^n k(x),\ k'\left(x^{2^n}\right) = 8^n k'(x),$$

for any $n \in \mathbb{N}$ and any $x \in G$. So

$$\begin{aligned}\left\|k(x) - k'(x)\right\| &= \frac{1}{8^n}\left\|k\left(x^{2^n}\right) - k'\left(x^{2^n}\right)\right\| \leq \frac{1}{8^n}\left\|k\left(x^{2^n}\right) - f\left(x^{2^n}\right)\right\| + \frac{1}{8^n}\left\|f\left(x^{2^n}\right) - k'\left(x^{2^n}\right)\right\| \\ &\leq \frac{1}{8^n}\left(\frac{1}{7}\delta\right).\end{aligned}$$

Therefore $k = k'$ □

Lemma 2.4. *Suppose that the couple (G, X) is CS. Let f, T and ϵ be same as those in Definition 2.1. Then T is unique and*

$$\|f(x) - T(x)\| \leq \frac{1}{14}\delta.$$

Moreover $\frac{1}{14}\delta$ is the best possible upper bound for the above inequality.

Proof. By Lemma 2.3, T is unique and

$$\|f(x) - T(x)\| \leq \frac{1}{14}\delta.$$

For the last assertion we consider the function $f(x) = \frac{1}{14}\delta$. □

Theorem 2.5. *Suppose that the couple $(G, \mathbb{C})$ is CS. Then for every complex Banach space X, the couple (G, X) is CS.*

Proof. Let $f : G \to X$ be a function satisfying (2.2) for all $x, y \in G$ and some $\delta \geq 0$. Let $\phi \in X^*$ where X^* denotes the dual space of X. Then the function $\phi o f : G \to \mathbb{C}$ satisfies the inequality (2.2). Indeed,

$$\left|\phi o f\left(x^2 y\right) + \phi o f\left(x^2 y^{-1}\right) - 2\phi o f(xy) - 2\phi o f\left(xy^{-1}\right) - 12\phi o f(x)\right| \leq \|\phi\|\delta.$$

Since the couple $(G, \mathbb{C})$ is CS, by Lemma 2.4, there exists a cubic function $g_\phi : G \to C$ such that

$$|\phi o f(x) - g_\phi(x)| \leq \frac{1}{14}\|\phi\|\delta. \tag{2.9}$$

From Lemma 2.3, the limit

$$h(x) = \lim_{n\to\infty} \frac{1}{8^n} f\left(x^{2^n}\right)$$

exists for all $x \in G$. Replacing x by x^{2^n} in (2.9), we get

$$\left|\phi o f\left(x^{2^n}\right) - g_\phi\left(x^{2^n}\right)\right| \le \frac{1}{14}\|\phi\|\delta.$$

But g_ϕ is cubic. Hence

$$\left|\frac{1}{8^n}\phi o f\left(x^{2^n}\right) - g_\phi(x)\right| \le \frac{1}{14}\left(\frac{1}{8^n}\right)\|\phi\|\delta.$$

Therefore

$$g_\phi(x) = \lim_{n\to\infty}\frac{1}{8^n}\phi o f\left(x^{2^n}\right) = \phi o h(x).$$

Moreover we have

$$\begin{aligned}\phi&\left(h\left(x^2y\right) + h\left(x^2y^{-1}\right) - 2h(xy) - 2h\left(xy^{-1}\right) - 12h(x)\right)\\ &= g_\phi\left(x^2y\right) + g_\phi\left(x^2y^{-1}\right) - 2g_\phi(xy) - 2g_\phi\left(xy^{-1}\right) - 12g_\phi(x) = 0,\end{aligned}$$

because g_ϕ is cubic. So h is a cubic mapping. By Lemma 2.3, we obtain that (G, X) is CS, and this completes the proof of the theorem. □

Definition 2.6. We say that a mapping $f : G \to X$ is a quasi-cubic mapping if there exists a nonnegative number δ such that

$$\left\|f\left(x^2y\right) + f\left(x^2y^{-1}\right) - 2f(xy) - 2f\left(xy^{-1}\right) - 12f(x)\right\| \le \delta,$$

for all $x, y \in G$. It is clear that the set of all quasi-cubic mappings from G into X is a real linear space relative to the ordinary operations. We denote it by $KC(G, X)$. The subspace of $KC(G, X)$ consisting of all cubic mappings will be denoted by $C(G, X)$.

Definition 2.7. The mapping $f : G \to X$ is said to be a pseudo-cubic mapping if it is a quasi-cubic mapping satisfying

$$f\left(x^n\right) = n^3 f(x),$$

for any $x \in G$ and any $n \in \mathbb{N}$. We denote the space of all pseudo-cubic mappings from G into X by $PC(G, X)$.

The space of all bounded mappings $f : G \to X$ will be denoted by $B(G, X)$.

Remark 2.8. The spaces $PC(G, \mathbb{R})$ and $C(G, \mathbb{R})$ will be denoted by $PC(G)$ and $C(G)$, respectively.

We recall that if n is an integer then a group G is said to be an n-Abelian group if

$$(ab)^n = a^n b^n,$$

for every $a, b \in G$.

Lemma 2.9. *Let $f \in KC(G, X)$. Then for any $k, m \in \mathbb{N}$, there exists $\delta_m > 0$ such that for each $x \in G$, the following relation*

$$\left\|\frac{1}{m^{3k}}f\left(x^{m^k}\right) - f(x)\right\| \le 2b_m, \tag{2.10}$$

holds, where $b_m = \frac{1}{m^3}\delta_m$.

Proof. Let f satisfies the inequality (2.2) for all $x, y \in G$ and some $\delta \geq 0$. Let $x \in G$. Substituting $y = e$ in the relation (2.2), we get

$$\left\| f\left(x^2\right) - 8f(x) \right\| \leq \frac{1}{2}\delta. \tag{2.11}$$

Replacing $y = x$ in (2.2), we see that

$$\left\| f\left(x^3\right) - 2f\left(x^2\right) - 11f(x) - 2f(e) \right\| \leq \delta. \tag{2.12}$$

Replacing $x = y = e$ in (2.2), we obtain

$$\|f(e)\| \leq \frac{1}{14}\delta. \tag{2.13}$$

So

$$\left\| f\left(x^3\right) - 27f(x) \right\| \leq \frac{15}{7}\delta. \tag{2.14}$$

We claim that for any integer $m \geq 1$, there exists $\delta_m > 0$ such that for each $x \in G$

$$\left\| f\left(x^m\right) - m^3 f(x) \right\| \leq \delta_m. \tag{2.15}$$

If we put $\delta_1 = \delta$, $\delta_2 = \frac{1}{2}\delta$ and $\delta_3 = \frac{15}{7}\delta$, then for $m \leq 3$, the assertion is easily verified. We prove the assertion for $m \geq 4$ by induction on m. Let $m \geq 4$ and suppose that (2.15) has been already verified for m. We prove it for $m + 1$. Putting $y = x^{m-1}$ in (2.2), we get

$$\left\| f\left(x^{m+1}\right) + f\left(x^{3-m}\right) - 2f\left(x^m\right) - 2f\left(x^{2-m}\right) - 12f(x) \right\| \leq \delta. \tag{2.16}$$

Replacing $x = e$ in (2.2), we obtain

$$\left\| f(y) + f\left(y^{-1}\right) + 12f(e) \right\| \leq \delta,$$

for all $y \in G$. So by (2.13), we get

$$\left\| f(y) + f\left(y^{-1}\right) \right\| \leq \frac{13}{7}\delta, \tag{2.17}$$

for all $y \in G$. Putting $y = x^{m-3}$ and $y = x^{m-2}$ in the last inequality respectively, we get

$$\left\| f\left(x^{m-3}\right) + f\left(x^{3-m}\right) \right\| \leq \frac{13}{7}\delta, \tag{2.18}$$

$$\left\| f\left(x^{m-2}\right) + f\left(x^{2-m}\right) \right\| \leq \frac{13}{7}\delta. \tag{2.19}$$

Moreover from the induction hypothesis we obtain the following relations

$$\left\| f\left(x^{m-3}\right) - (m-3)^3 f(x) \right\| \leq \delta_{m-3}, \tag{2.20}$$

$$\left\| f\left(x^{m-2}\right) - (m-2)^3 f(x) \right\| \le \delta_{m-2}, \tag{2.21}$$

$$\left\| 2f\left(x^{m}\right) - 2m^3 f(x) \right\| \le 2\delta_{m}. \tag{2.22}$$

It follows from the relations (2.16), (2.18), (2.19), (2.20), (2.21) and (2.22) that

$$\left\| f\left(x^{m+1}\right) - (m+1)^3 f(x) \right\| \le \frac{46}{7}\delta + \delta_{m-3} + 2\delta_{m-2} + 2\delta_m.$$

Letting

$$\delta_{m+1} = \frac{46}{7}\delta + \delta_{m-3} + 2\delta_{m-2} + 2\delta_m,$$

we get (2.15).
Now we prove (2.10). The proof is by induction on k. If $k = 1$, then the assertion is clearly true by (2.15). Let $k > 1$. From (2.15), we have

$$\left\| \frac{1}{m^3} f\left(x^{m}\right) - f(x) \right\| \le b_m. \tag{2.23}$$

Replacing x by x^m in the last inequality, we obtain

$$\left\| \frac{1}{m^3} f\left(x^{m^2}\right) - f\left(x^{m}\right) \right\| \le b_m. \tag{2.24}$$

Hence we have

$$\left\| \frac{1}{m^6} f\left(x^{m^2}\right) - \frac{1}{m^3} f\left(x^{m}\right) \right\| \le \frac{1}{m^3} b_m. \tag{2.25}$$

So we get

$$\left\| \frac{1}{m^6} f\left(x^{m^2}\right) - f(x) \right\| \le b_m \left(1 + \frac{1}{m^3}\right). \tag{2.26}$$

Letting $x = x^m$ in the last inequality, we obtain

$$\left\| \frac{1}{m^6} f\left(x^{m^3}\right) - f\left(x^{m}\right) \right\| \le b_m \left(1 + \frac{1}{m^3}\right). \tag{2.27}$$

Hence

$$\left\| \frac{1}{m^9} f\left(x^{m^3}\right) - f(x) \right\| \le b_m \left(1 + \frac{1}{m^3} + \frac{1}{m^6}\right). \tag{2.28}$$

Continuing in this manner, we get the following inequality

$$\left\| \frac{1}{m^{3k}} f\left(x^{m^k}\right) - f(x) \right\| \le b_m \left(1 + \frac{1}{m^3} + \frac{1}{m^6} + \cdots + \frac{1}{m^{3(k-1)}}\right) \le 2b_m.$$

This completes the proof of the theorem. □

Lemma 2.10. *If $f \in PC(G, X)$, then*

1. $f(e) = 0$,

2. *$f\left(x^{-n}\right) = -n^3 f(x)$ for any $x \in G$ and any $n \in \mathbb{N}$,*
3. *if $y \in G$ is an element of finite order then $f(y) = 0$,*
4. *if f is a bounded function on G then $f \equiv 0$.*

Proof. 1. $f(e) = f\left(e^n\right) = n^3 f(e)$ for any $n \in \mathbb{N}$. Hence $f(e) = 0$.

2. It follows from (2.17) that

$$\left\| f\left(x^k\right) + f\left(x^{-k}\right) \right\| \leq \frac{13}{7}\delta,$$

or

$$\left\| f(x) + f\left(x^{-1}\right) \right\| \leq \frac{13}{7k^3}\delta,$$

for any $x \in G$ and any $k \in \mathbb{N}$. So

$$f(x) + f(x^{-1}) = 0,$$

for any $x \in G$. Therefore we have

$$f\left(x^{-n}\right) = -f\left(x^n\right) = -n^3 f(x),$$

for any $n \in \mathbb{N}$ and any $x \in G$.

3. There exists $n \in \mathbb{N}$ such that $y^{-n} = e$. So we get $-n^3 f(y) = 0$. Hence $f(y) = 0$.
4. Let $y \in G$. We have $\|f\left(y^n\right)\| \leq c$ for some $c > 0$ and any $n \in \mathbb{N}$. Hence $\|f(y)\| \leq \frac{c}{n^3}$ for any $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we get $f(y) = 0$. This completes the proof of the lemma.

□

Lemma 2.11. *Let $f \in KC(G, X)$. Then the sequence $\left(\frac{1}{m^{3k}} f\left(x^{m^k}\right)\right)_k$ is a Cauchy sequence for any $x \in G$ and any $m \in \mathbb{N}$.*

Proof. Let $x \in G$ and $n, m, k \in \mathbb{N}$. It follows from Lemma 2.9 that

$$\left\| \frac{1}{m^{3k}} f\left(x^{m^{n+k}}\right) - f\left(x^{m^n}\right) \right\| \leq 2b_m.$$

So

$$\left\| \frac{1}{m^{3(n+k)}} f\left(x^{m^{n+k}}\right) - \frac{1}{m^{3n}} f\left(x^{m^n}\right) \right\| \leq \frac{1}{m^{3n}} 2b_m.$$

From the last inequality, we conclude that the sequence $\left(\frac{1}{m^{3k}} f\left(x^{m^k}\right)\right)_k$ is a Cauchy sequence. □

Definition 2.12. From Lemma 2.11, we conclude that the sequence $\left(\frac{1}{m^{3k}} f\left(x^{m^k}\right)\right)_k$ has a limit. We denote it by $f_m(x)$. Therefore

$$f_m(x) := \lim_{k \to \infty} \frac{1}{m^{3k}} f\left(x^{m^k}\right).$$

Lemma 2.13. *Let $f \in KC(G, X)$. Then for any $x \in G$ and any $n, m \in \mathbb{N}$, we have*

$$f_m\left(x^{m^n}\right) = m^{3n} f_m(x).$$

Proof. We have

$$f_m\left(x^{m^n}\right)=\lim_{k\to\infty}\frac{1}{m^{3k}}f\left(x^{m^{n+k}}\right)=m^{3n}\lim_{k\to\infty}\frac{1}{m^{3(n+k)}}f\left(x^{m^{n+k}}\right)=m^{3n}f_m(x),$$

for any $x\in G$ and any $n,m\in\mathbb{N}$. □

Lemma 2.14. *Let $f\in KC(G,X)$. Then $f_m\in KC(G,X)$ for all $m\in\mathbb{N}$.*

Proof. Fix $m\in\mathbb{N}$. Let $f:G\to X$ satisfies the inequality (2.2) for all $x,y\in G$ and some $\delta\geq 0$. It follows from Lemma 2.9 that for each $x\in G$

$$\|f_m(x)-f(x)\|\leq 2b_m.$$

Hence

$$\begin{aligned}&\left\|f_m\left(x^2y\right)+f_m\left(x^2y^{-1}\right)-2f_m(xy)-2f_m\left(xy^{-1}\right)-12f_m(x)\right\|\leq\left\|f_m\left(x^2y\right)-f\left(x^2y\right)\right\|\\&+\left\|f_m\left(x^2y^{-1}\right)-f\left(x^2y^{-1}\right)\right\|+2\left\|f_m(xy)-f(xy)\right\|+2\left\|f_m\left(xy^{-1}\right)-f\left(xy^{-1}\right)\right\|\\&+12\left\|f_m(x)-f(x)\right\|+\left\|f\left(x^2y\right)+f\left(x^2y^{-1}\right)-2f(xy)-2f\left(xy^{-1}\right)-12f(x)\right\|\\&\leq 36b_m+\delta.\end{aligned}$$

So $f_m\in KC(G,X)$. □

Lemma 2.15. *Let $f\in KC(G,X)$. Then for any positive integer $m\geq 2$, we have $f_2=f_m$.*

Proof. Consider the function ϕ defined by

$$\phi(x):=\lim_{k\to\infty}\frac{1}{m^{3k}}f_2\left(x^{m^k}\right).$$

Note that $\phi\in KC(G,X)$. Let $x\in G$. From Lemma 2.13, we conclude that

$$\phi\left(x^{m^k}\right)=m^{3k}\phi(x),\ \phi\left(x^{2^k}\right)=8^k\phi(x),\tag{2.29}$$

for any $k\in\mathbb{N}$. It follows from Lemma 2.9 that there exists $c>0$ such that

$$\|f_2(x)-\phi(x)\|\leq c.\tag{2.30}$$

Replacing x by x^{2^k} in (2.30), we get

$$\left\|f_2\left(x^{2^k}\right)-\phi\left(x^{2^k}\right)\right\|\leq c.$$

So

$$\|f_2(x)-\phi(x)\|\leq\frac{1}{8^k}c,$$

for any $k\in\mathbb{N}$. Hence

$$\phi(x)=f_2(x).\tag{2.31}$$

Moreover

$$\|f_m(x)-\phi(x)\|\leq\|f_m(x)-f(x)\|+\|f(x)-f_2(x)\|+\|f_2(x)-\phi(x)\|.$$

Hence

$$\|f_m(x) - \phi(x)\| \leq d, \tag{2.32}$$

for some $d > 0$. Therefore similar to the proof of the relation (2.31), we obtain

$$\phi(x) = f_m(x). \tag{2.33}$$

This completes the proof of the lemma. □

Definition 2.16. We denote the function ϕ introduced in Lemma 2.15 by $\hat{f}$. So for any $f \in KC(G, X)$ the function $\hat{f}$ is defined as

$$\hat{f}(x) := \lim_{k\to\infty} \frac{1}{8^k} f\left(x^{2^k}\right). \tag{2.34}$$

Corollary 2.17. *$\hat{f}(x^n) = n^3\hat{f}(x)$, for any $x \in G$ and any $n \in \mathbb{N}$.*

Proof. Let $x \in G$ and $2 \leq n \in \mathbb{N}$. Then $\hat{f}\,(x^n) = f_n\,(x^n) = n^3 f_n(x) = n^3\hat{f}(x)$. □

Theorem 2.18. *$KC(G, X) = PC(G, X) \oplus B(G, X)$.*

Proof. It is easy to see that $PC(G, X)$ and $B(G, X)$ are subspaces of $KC(G, X)$. Let us show that

$$PC(G, X) \bigcap B(G, X) = \{0\}.$$

Let $x \in G$ and $n \in \mathbb{N}$. If

$$f \in PC(G, X) \bigcap B(G, X),$$

then for some $c_f > 0$ we have $\|f\,(x^n)\,\| \leq c_f$. Therefore

$$n^3\|f(x)\| \leq c_f,$$

or

$$\|f(x)\| \leq \frac{1}{n^3} c_f.$$

Hence $f(x) = 0$. Let f be an arbitrary element from $KC(G, X)$, then from Corollary 2.17, we conclude that

$$\hat{f} \in PC(G, X).$$

Moreover

$$\hat{f}(x) = f_2(x).$$

Therefore we have

$$\|f(x) - \hat{f}(x)\| = \|f(x) - f_2(x)\|.$$

It follows from Lemma 2.9 that

$$f - \hat{f} \in B(G, X).$$

□

Theorem 2.19. *The cubic functional equation* (2.1) *is stable for the pair* (G, X) *if and only if* $PC(G, X) = C(G, X)$.

Proof. It is clear that $C(G, X)$ is a subspace of $PC(G, X)$. If cubic functional equation (2.1) is stable for the pair (G, X), then

$$PC(G, X) = C(G, X),$$

because if there exists

$$f \in PC(G, X) - C(G, X),$$

then from the assumption we conclude that there exists $g \in C(G, X)$ such that for some nonnegative number δ we have

$$\|f(x) - g(x)\| \leq \delta,$$

for any $x \in G$. So

$$\left\| f(x) - g(x) \right\| = \frac{1}{8^n} \left\| f\left(x^{2^n}\right) - g\left(x^{2^n}\right) \right\| \leq \frac{1}{8^n}\delta,$$

for any $x \in G$ and any $n \in \mathbb{N}$. Hence $f = g$. Thus we come to a contradiction with the assumption about f. Conversely if $PC(G, X) = C(G, X)$ and $f \in KC(G, X)$ then from Theorem 2.18, we conclude that $f = g + h$ where $g \in C(G, X)$ and $h \in B(G, X)$. So

$$f - g \in B(G, X).$$

□

Theorem 2.20. *Let* X, Y *be Banach spaces over reals. Then the cubic functional equation* (2.1) *is stable for the pair* (G, X) *if and only if it is stable for the pair* (G, Y).

Proof. We prove that the cubic functional equation (2.1) is stable for the pair (G, X) if and only if it is stable for the pair $(G, \mathbb{R})$ where X is a Banach space and $\mathbb{R}$ is the set of reals.
Let the cubic functional equation (2.1) be stable for the pair (G, X). Suppose that it is not stable for the pair $(G, \mathbb{R})$. Then there is a function f such that

$$f \in PC(G, \mathbb{R}) - C(G, \mathbb{R}).$$

So for some $\delta \geq 0$, we have

$$\left| f\left(x^2y\right) + f\left(x^2y^{-1}\right) - 2f(xy) - 2f\left(xy^{-1}\right) - 12f(x) \right| \leq \delta,$$

for each $x, y \in G$. Choose $e \in X$ such that $\|e\| = 1$. Let $g : G \to X$ be a mapping defined by the formula

$$g(x) := f(x)e.$$

It is easy to see that

$$g \in PC(G, X) - C(G, X).$$

So we obtain a contradiction.
Now suppose that the cubic functional equation (2.1) is stable for the pair $(G, \mathbb{R})$. So

$$PC(G, \mathbb{R}) = C(G, \mathbb{R}).$$

Let there exists a mapping $f : G \to X$ such that

$$f \in PC(G, X) - C(G, X).$$

So there are $x, y \in G$, such that

$$f\left(x^2y\right) + f\left(x^2y^{-1}\right) - 2f(xy) - 2f\left(xy^{-1}\right) - 12f(x) \neq 0.$$

Therefore by Hahn-Banach Theorem, we conclude that there is $\phi \in X^*$ such that

$$\phi\left(f\left(x^2y\right) + f\left(x^2y^{-1}\right) - 2f(xy) - 2f\left(xy^{-1}\right) - 12f(x)\right) \neq 0.$$

We prove that $\phi o f \in PC(G, \mathbb{R}) - C(G, \mathbb{R})$.
Indeed, if δ is a nonnegative number such that for any $x, y \in G$, the inequality

$$\left\| f\left(x^2y\right) + f\left(x^2y^{-1}\right) - 2f(xy) - 2f\left(xy^{-1}\right) - 12f(x) \right\| \leq \delta,$$

holds, then

$$\left| \phi o f\left(x^2y\right) + \phi o f\left(x^2y^{-1}\right) - 2\phi o f(xy) - 2\phi o f\left(xy^{-1}\right) - 12\phi o f(x) \right| \leq \delta\|\phi\|.$$

It is evident that

$$\phi o f\left(x^n\right) = n^3\phi o f(x),$$

for any $x \in G$ and any $n \in \mathbb{N}$. So

$$\phi o f \in PC(G, \mathbb{R}) - C(G, \mathbb{R}).$$

This contradiction completes the proof of the theorem. □

Due to the last theorem we may simply say that the cubic functional equation (2.1) is stable or not stable on a group G.

Definition 2.21. We shall say that an element x of a group G is periodic if there are $m, n \in \mathbb{N}$ such that $m \neq n$ and $x^m = x^n$. The group G is said to be periodic if every element of G is periodic.

Corollary 2.22. *The cubic functional equation* (2.1) *is stable on any periodic group.*

Proof. Let $f \in PC(G, X)$ and $x \in G$. Then there are $m, n \in \mathbb{N}$ such that $m \neq n$ and $f\left(x^m\right) = f\left(x^n\right)$. So $m^3f(x) = n^3f(x)$. Hence $\left(m^3 - n^3\right)f(x) = 0$, and thus $f(x) = 0$. □

Now we present our main result.

Theorem 2.23. *Let $n \in \mathbb{N}$ and G be an n-Abelian group. Then the cubic functional equation* (2.1) *is stable on group G.*

Proof. We show that

$$PC(G) = C(G).$$

Let $f \in PC(G)$ and $\delta \geq 0$ be such that for any $x, y \in G$, the inequality

$$\left| f\left(x^2y\right) + f\left(x^2y^{-1}\right) - 2f(xy) - 2f\left(xy^{-1}\right) - 12f(x) \right| \leq \delta, \tag{2.35}$$

holds. Let a, b be arbitrary elements of G. We show that

$$f\left(a^2b\right)+f\left(a^2b^{-1}\right)-2f(ab)-2f\left(ab^{-1}\right)-12f(a)=0.$$

We have

$$(ab)^n=a^nb^n.$$

So for any $m\in\mathbb{N}$, we have

$$(ab)^{n^m}=a^{n^m}b^{n^m}. \tag{2.36}$$

We prove this by induction on m. If $m=1$, the above relation is true.
Suppose that (2.36), is true for m. Then we have

$$(ab)^{n^{m+1}}=\left((ab)^{n^m}\right)^n=\left(a^{n^m}b^{n^m}\right)^n=a^{n^{m+1}}b^{n^{m+1}}.$$

So for any $m\in\mathbb{N}$, we get

$$\begin{aligned}&n^{3m}\left|f\left(a^2b\right)+f\left(a^2b^{-1}\right)-2f(ab)-2f\left(ab^{-1}\right)-12f(a)\right|\\&=\left|f\left(\left(a^{n^m}\right)^2b^{n^m}\right)+f\left(\left(a^{n^m}\right)^2\left(b^{n^m}\right)^{-1}\right)-2f\left(a^{n^m}b^{n^m}\right)-2f\left(a^{n^m}\left(b^{n^m}\right)^{-1}\right)-12f\left(a^{n^m}\right)\right|\le\delta.\end{aligned}$$

Hence

$$\left|f\left(a^2b\right)+f\left(a^2b^{-1}\right)-2f(ab)-2f\left(ab^{-1}\right)-12f(a)\right|\le\frac{1}{n^{3m}}\delta,$$

for any $m\in\mathbb{N}$. Therefore, we have $f\in C(G)$ and this completes the proof of the theorem. □

It is well known that every Abelian group is an n-Abelian group for any $n\in\mathbb{N}$. Thus we get another version of Lemma 2.2 as a result.

Corollary 2.24. *The cubic functional equation* (2.1) *is stable on any Abelian group.*

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