Available online at www.vonneumann-publishing.com/fatma
Func. Anal.-TMA 3 (2017), 15–25
Research Article

Functional Analysis: Theory, Methods & Applications
Journal Homepage: http://vonneumann-publishing.com/fatma

Stabilities and counter-examples of mixed Euler-Lagrange k-cubic-quartic functional equation in quasi-β-normed spaces

J. M. Rassias[a], B. V. Senthil Kumar[b,*], S. Sabarinathan[c]

[a]Pedagogical Department E.E., Section of Mathematics and Informatics, National and Capodistrian University of Athens, 4, Agamemnonos Str., Aghia Paraskevi, Athens, Attikis 15342, Greece.

[b]Department of Mathematics, C. Abdul Hakeem College of Engineering & Technology, Melvisharam-632 509, Tamil Nadu, India.

[c]Department of Mathematics, SRM Institute of Science & Technology, Kattankulthur-603 203, Tamil Nadu, India.

Communicated by J. Chmielinski

Abstract

The intention of this study is to introduce a new mixed Euler-Lagrange k-cubic-quartic functional equation and then to solve it for general solution. We study its various stabilities in quasi-β-normed spaces using fixed point technique, as well. We also provide counter-examples to show that the above equation is not stable for singular cases.

Keywords: Cubic mapping, quartic mapping, quasi-β-normed space, (β, p)-Banach space, generalized Ulam-Hyers stability.
2010 MSC: 39B82, 39B72.

1. Introduction & Preliminaries

The starting point of studying the stability of functional equations seems to be the famous talk of Ulam [21] in 1940, in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms.

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) <$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$.

*Corresponding author
Email addresses: jrassias@primedu.uoa.gr (J. M. Rassias), bvskumarmaths@gmail.com (B. V. Senthil Kumar), ssabarimaths@gmail.com (S. Sabarinathan)

Received 2018-01-24

The first partial answer, in the case of Cauchy equation in Banach spaces, to Ulam question was given by Hyers [8]. Later, the result of Hyers was generalized in various forms by Aoki [1], T. M. Rassias [10], J. M. Rassias ([11], [12], [13]) and Gávruta [7]. Since then, the stability problems of several functional equations have been extensively investigated by a number of mathematicians [2], [5], [14], [15], [16], [17], [18], [25].

The motivation for studying cubic and quartic functional equations came from the fact that recently polynomial equations have found applications in approximate checking, self-testing and self-correcting of computer programs to compute certain polynomials, computational geometry and all related fields such as computer graphics, computer-aided design, computer-aided manufacturing and optics.

In this paper, we introduce a new mixed Euler-Lagrange k-cubic-quartic functional equation

$$\begin{aligned} f(x+ky)+f(kx+y)+f(x-ky)+f(y-kx) \\ &= k^2[2f(x+y)+f(x-y)+f(y-x)]-2(k^4-1)[f(x)+f(y)] \\ &\quad + \frac{1}{4}k^2(k^2-1)[f(2x)+f(2y)], \end{aligned} \tag{1.1}$$

where k is a real number with $k \neq 0, \pm 1$. We attain the general solution of equation (1.1) and study its various stabilities in quasi-β-normed spaces. We also provide counter-examples to show that the equation (1.1) is not stable for singular cases.

Here, we summonup some fundamental notions related to m-additive symmetric mappings, generalized polynomial and quasi-β-normed spaces. For further details of m-additive symmetric mappings, one may refer ([3], [19], [20], [23], [24]).

Let X and Y be real vector spaces. A function $g: X \to Y$ is said to be additive if $g(u+v) = g(u)+g(v)$ for all $u, v \in X$. It is easy to see that $g(ru) = rg(u)$ for all $u \in X$ and all $r \in \mathbb{Q}$ (the set of rational numbers).

Let $m \in \mathbb{N}$ (the set of natural numbers). A function $H : X^m \to Y$ is called m-additive if it is additive in each of its variables. A function H_m is called symmetric if $H_m(u_1, u_2, \ldots, u_m) = H_m\left(u_{\pi(1)}, u_{\pi(2)}, \ldots, u_{\pi(m)}\right)$ for every permutation $\{\pi(1), \pi(2), \ldots, \pi(m)\}$ of $\{1, 2, \ldots, m\}$. If $H_m(u_1, u_2, \ldots, u_m)$ is m-additive symmetric map, then $H^m(u)$ will denote the diagonal $H_m(u, u, \ldots, u)$ for $u \in X$ and note that $H^m(ru) = r^m H^m(u)$ whenever $u \in X$ and $r \in \mathbb{Q}$. Such a function $H(x)$ will be called a monomial function of degree m (assuming $H^m \not\equiv 0$). Furthermore the resulting function after substitution $u_1 = u_2 = \cdots = u_\ell = u$ and $u_{\ell+1} = u_{\ell+2} = \cdots = u_m = v$ in $H_m(u_1, u_2, \ldots, u_m)$ will be denoted by $H^{\ell, m-\ell}(u, v)$.

A function $q : X \to Y$ is called a generalized polynomial function of degree $m \in \mathbb{N}$ provided that there exist $H^0(u) = H^0 \in Y$ and i-additive symmetric functions $H_i : X^i \to Y$ (for $1 \leq i \leq m$) such that $q(u) = \sum\limits_{i=0}^{m} H^i(u)$, for all $u \in X$ and $H^m \not\equiv 0$.

For $A : X \to Y$, let Δ_h be the difference operator defined as follows:

$$\Delta_h A(u) = A(u+h) - A(u),$$

for $h \in X$. Furthermore, let $\Delta_h^0 A(u) = A(u)$, $\Delta_h^1 = \Delta_h A(u)$ and $\Delta_h \circ \Delta_h^m A(u) = \Delta_h^{n+1} A(u)$ for all $m \in \mathbb{N}$ and all $h \in X$. Here $\Delta_h \circ \Delta_h^m$ denotes the composition of the operators Δ_h and Δ_h^m. For any given $m \in \mathbb{N}$, the functional equation $\Delta_h^{m+1} A(u) = 0$ for all $u, h \in X$ is well studied. In explicit form the last functional equation can be written as

$$\Delta_h^{m+1} A(u) = \sum_{j=0}^{m+1} (-1)^{m+1-j} \binom{m+1}{j} A(u+jh) = 0.$$

The following theorem was proved by Mazur and Orlicz, and in greater generality by Djoković (see [4]).

Theorem 1.1. *Let X and Y be real vector spaces, $n \in \mathbb{N}$ and $A : X \to Y$, then the following are equivalent.*

(1) $\Delta_h^{m+1} A(u) = 0$ *for all $u, h \in X$.*

(2) $\Delta_{u_1,\ldots,u_{m+1}}A(u_0) = 0$ *for all* $u_0, u_1, \ldots, u_{m+1} \in X$.

(3) $A(u) = H^m(u) + H^{m-1}(u) + \cdots + H^2(u) + H^1(u) + H^0(u)$ *for all* $u \in X$*, where* $H^0(u) = H^0$ *is an arbitrary element of* Y *and* $H^i(u)(i = 1, 2, \ldots, m)$ *is the diagonal of an* i*-additive symmetric function* $H_i : X^i \to Y$.

We fix a real number β with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{F}$ be a linear space over $\mathbb{K}$. A quasi-β-norm $\|\cdot\|$ is a real-valued function on $\mathcal{F}$ satisfying the following:

(i) $\|u\| \geq 0$ for all $u \in \mathcal{F}$ and $\|u\| = 0$ if and only if $u = 0$.

(ii) $\|\lambda u\| = |\lambda|^\beta \cdot \|u\|$ for all $\lambda \in \mathbb{K}$ and all $u \in \mathcal{F}$.

(iii) There is a constant $K \geq 1$ such that $\|u + v\| \leq K(\|u\| + \|v\|)$, for all $u, v \in \mathcal{F}$.

The pair $(\mathcal{F}, \|\cdot\|)$ is called quasi-β-normed space if $\|\cdot\|$ is a quasi-β-norm on $\mathcal{F}$. The smallest possible K is called the modulus of concavity of $\|\cdot\|$. A quasi-β-Banach space is a complete quasi-β-normed space.

A quasi-β-norm $\|\cdot\|$ is called a (β, p)-norm $(0 < p \leq 1)$ if $\|u + v\|^p \leq \|u\|^p + \|v\|^p$, for all $u, v \in \mathcal{F}$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space.

2. General solution of functional equation (1.1)

In this section, let us assume X and Y to be vector spaces. In the upcoming results, we acquire the general solution of mixed Euler-Lagrange k-cubic-quartic functional equation (1.1).

Theorem 2.1. *An odd function* $f : X \to Y$ *is a solution of the functional equation* (1.1) *if and only if* f *is of the form* $f(x) = A^3(x)$ *for all* $x \in X$*, where* $A^3(x)$ *is the diagonal of the* 3*-additive symmetric map* $A_3 : X^3 \to Y$.

Proof. Assume that f satisfies the functional equation (1.1). Replacing (x, y) by $(0, 0)$, one finds that $f(0) = 0$. Replacing (x, y) by $(x, 0)$ in (1.1) and using oddness of f, we get $f(2x) = 2^3 f(x)$. Therefore $f(x) = A^3(x) + A^2(x) + A^1(x) + A^0(x)$ for all $x \in X$, where $A^0(x) = A^0$ is an arbitrary element of Y, and $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i : X^i \to Y$ for $i = 1, 2, 3$. By $f(0) = 0$ and oddness of f, we get $A^0(x) = A^0 = 0$. Thus we have $A^2(x) = 0$. It follows that $f(x) = A^3(x) + A^1(x)$ and $A^n(rx) = r^n A^n(x)$ then

$$2^3(A^3(x) + A^1(x)) = 2^3 A^3(x) + 2A^1(x).$$

Moreover $A^1(x) = 0$. Therefore $f(x) = A^3(x)$ for all $x \in X$.

Conversely assume that $f(x) = A^3(x)$. From $A^3(x + y) = A^3(x) + A^3(y) + 3A^{2,1}(x, y) + 3A^{1,2}(x, y)$, $A^3(rx) = r^3 A^3(x)$, $A^{2,1}(x, ry) = rA^{2,1}(x, y)$, $A^{1,2}(x, ry) = r^2 A^{1,2}(x, y)$ $(x, y \in X, r \in \mathbb{Q})$, we see that f satisfies (1.1), which completes the proof. □

Theorem 2.2. *An even function* $f : X \to Y$ *with* $f(2x) = 16f(x)$ *is a solution of the functional equation* (1.1) *if and only if* f *is of the form* $f(x) = A^4(x)$ *for all* $x \in X$*, where* $A^4(x)$ *is the diagonal of the* 4*-additive symmetric map* $A_4 : X^4 \to Y$.

Proof. Assume that f satisfies the functional equation (1.1). Replacing (x, y) by $(0, 0)$, one finds that $f(0) = 0$. Switching (x, y) to $(x, 0)$ in (1.1) and using evenness of f, we get $f(kx) = k^4 f(x)$. Therefore $f(x) = A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x)$ for all $x \in X$, where $A^0(x) = A^0$ is an arbitrary element of Y, and $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i : X^i \to Y$ for $i = 1, 2, 3$. By $f(0) = 0$ and eveness of f, we get $A^0(x) = A^0 = 0$. Thus we have $A^3(x) = A^1(x) = 0$. It follows that $f(x) = A^4(x) + A^2(x)$ and $A^n(rx) = r^n A^n(x)$ then

$$2^4(A^4(x) + A^2(x)) = 2^4 A^4(x) + 2^2 A^2(x).$$

Moreover $A^2(x) = 0$. Therefore $f(x) = A^4(x)$ for all $x \in X$. Conversely assume that $f(x) = A^4(x)$. From $A^4(x+y) = A^4(x)+A^4(y)+4A^{3,1}(x, y)+6A^{2,2}(x, y)+4A^{1,3}(x)$, $A^4(rx) = r^4 A^4(x)$, $A^{3,1}(x, ry) = rA^{3,1}(x, y)$, $A^{2,2}(x, ry) = r^2 A^{2,2}(x, y)$, $A^{1,3}(x, ry) = r^3 A^{1,3}(x, y)$ $(x, y \in X, r \in \mathbb{Q})$, which completes the proof. □

3. Various stabilities of functional equation (1.1)

Throughout this section, we assume that X is a linear space and Y is a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_Y$. Let K be the modulus of concavity of $\|\cdot\|_Y$. For notational convenience, we define the difference operator for a given mapping $f : X \to Y$ as

$$\begin{aligned} D_k f(x,y) = {} & f(x+ky) + f(kx+y) + f(x-ky) + f(y-kx) \\ & - k^2[2f(x+y) + f(x-y) + f(y-x)] + 2(k^4-1)[f(x)+f(y)] \\ & - \frac{1}{4}k^2(k^2-1)[f(2x)+f(2y)], \end{aligned}$$

for all $x, y \in \mathcal{X}$.

Lemma 3.1. (see [22]). *Let $i \in \{-1, 1\}$ be fixed, $s, a \in \mathbb{N}$ with $a \geq 2$ and $\Psi : X \to [0, \infty)$ be a function such that there exists an $L < 1$ with $\Psi\left(a^i x\right) \leq a^{is\beta} L \Psi(x)$ for all $x \in X$. Let $f : X \to Y$ be a mapping satisfying*

$$\|f(ax) - a^s f(x)\|_Y \leq \Psi(x), \tag{3.1}$$

for all $x \in X$, then there exists a uniquely determined mapping $F : X \to Y$ such that $F(ax) = a^s F(x)$ and

$$\|f(x) - F(x)\|_Y \leq \frac{1}{a^{s\beta}\left|1 - L^i\right|} \Psi(x), \tag{3.2}$$

for all $x \in X$.

By applying the above Lemma 3.1, we investigate various stabilities of equation (1.1) in quasi-β-normed spaces.

Theorem 3.2. *Let $i \in \{-1, 1\}$ be fixed. Let $\phi : X \times X \to [0, \infty)$ be a function such that there exists an $L < 1$ with $\phi\left(2^i x, 2^i y\right) \leq 2^{3i\beta} L \phi(x, y)$ for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\|D_k f(x,y)\|_Y \leq \phi(x,y), \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying equation (1.1) and

$$\|f(x) - C(x)\|_Y \leq \frac{1}{2^{3\beta}\left|1 - L^i\right|} \Psi(x), \tag{3.4}$$

for all $x \in X$, where

$$\Psi(x) = \frac{4^\beta K}{k^{2\beta}(k^2-1)^\beta}\left[\phi(x,0) + \frac{\phi(0,0)}{2^\beta (k^2-1)^\beta}\right].$$

Proof. Substituting $x = y = 0$ in (3.3), we get

$$\|f(0)\|_Y \leq \frac{2^\beta}{7^\beta k^{2\beta}(k^2-1)^\beta} \phi(0,0). \tag{3.5}$$

Now, replacing (x, y) by $(0, x)$ in (3.3) and using oddness of f, we obtain

$$\left\| \frac{k^2(k^2-1)}{4}[f(2x) - 2^3 f(x)] - \frac{7k^2+8}{4} f(0) \right\|_Y \leq \phi(x,0), \tag{3.6}$$

for all $x \in X$. Using (3.5) in (3.6), one finds

$$\left\|f(2x) - 2^3 f(x)\right\|_Y \leq \Psi(x), \tag{3.7}$$

for all $x \in X$. By Lemma (3.1), there exists a unique mapping $C : X \to Y$ such that $C(2x) = 2^3 C(x)$ and

$$\|f(x) - C(x)\|_Y \leq \frac{1}{2^{3\beta}\left|1 - L^i\right|}\Psi(x),$$

for all $x \in X$. Let us show that C is a cubic mapping. By (3.3), we have

$$\begin{aligned}\left\|\frac{1}{2^{3in}}D_k f\left(2^{in}x, 2^{in}y\right)\right\|_Y &\leq 2^{-3in\beta}\phi\left(2^{in}x, 2^{in}y\right)\\ &\leq 2^{-3in\beta}\left(2^{3i\beta}L\right)^n \phi(x,y)\\ &= L^n\phi(x,y),\end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. So $\|D_k C(x,y)\|_Y = 0$ for all $x, y \in X$, which implies that the mapping $C : X \to Y$ is cubic. □

The following corollaries are direct outcomes of Theorem 3.2 pertinent to stability involving sum of powers of norms and mixed product-sum of powers of norms.

Corollary 3.3. *Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, and let Y be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_Y$. Let c_1, a be positive numbers with $a \neq \frac{3\beta}{r}$ and $f : X \to Y$ be an odd mapping satisfying*

$$\|D_k f(x,y)\|_Y \leq c_1\left(\|x\|_X^a + \|y\|_X^a\right),$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying equation (1.1) *and*

$$\|f(x) - C(x)\|_Y \leq \begin{cases} \frac{c_1 K 4^\beta}{k^{2\beta}(k^2-1)^\beta(2^{3\beta}-2^{ar})}\|x\|_X^a, & a \in \left(0, \frac{3\beta}{r}\right) \\ \\ \frac{c_1 K 4^\beta 2^{ar}}{k^{2\beta}(k^2-1)^\beta 2^{3\beta}(2^{ar}-2^{3\beta})}\|x\|_X^a, & a \in \left(\frac{3\beta}{r}, \infty\right), \end{cases}$$

for all $x \in X$, where

Proof. The proof is obtained by taking $\phi(x,y) = c_1\left(\|x\|_X^a + \|y\|_X^a\right)$, for all $x, y \in X$ and $L = \frac{2^{ar}}{2^{3\beta}}$ in Theorem 3.2. □

Corollary 3.4. *Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, and let Y be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_Y$. Let c_2, r, s be positive numbers with $a = r + s \neq \frac{11\beta}{\alpha}$ and $f : X \to Y$ be an odd mapping satisfying*

$$\|D_k f(x,y)\|_Y \leq c_2\left[\|x\|_X^r\|y\|_X^s + \left(\|x\|_X^{r+s} + \|y\|_X^{r+s}\right)\right],$$

for all non-zero $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ satisfying the equation (1.1) *and*

$$\|f(x) - C(x)\|_Y \leq \begin{cases} \frac{2c_2 K 4^\beta}{k^{2\beta}(k^2-1)^\beta(2^{3\beta}-2^{2ar})}\|x\|_X^a, & a \in \left(0, \frac{3\beta}{2r}\right) \\ \\ \frac{2c_2 K 4^\beta 2^{2ar}}{2^{3\beta}k^{2\beta}(k^2-1)^\beta(2^{3\beta}-2^{2ar})}\|x\|_X^a, & a \in \left(\frac{3\beta}{2r}, \infty\right), \end{cases}$$

for all $x \in X$.

Proof. By taking $\phi(x,y) = c_2\left[\|x\|_X^r\|y\|_X^s + \left(\|x\|_X^{r+s} + \|y\|_X^{r+s}\right)\right]$, for all $x, y \in X$ and $L = \frac{2^{2ar}}{2^{3\beta}}$ in Theorem 3.2, we arrive at the desired results. □

The ensuing theorem includes the investigation of generalized Ulam-Hyers stability of equation (1.1) in quasi-β-normed spaces. Even though the proof is similar to Theorem 3.2, for the sake of completeness, we present the entire proof of theorem.

Theorem 3.5. *Let $i \in \{-1, 1\}$ be fixed. Let $\phi : X \times X \to [0, \infty)$ be a function such that there exists an $L < 1$ with $\phi\left(2^i x, 2^i y\right) \leq 2^{4i\beta} L\phi(x, y)$ for all $x, y \in X$. Let $f : X \to Y$ be an even mapping with the condition $f(2x) = 2^4 f(x)$ satisfying*

$$\|D_u f(x, y)\|_Y \leq \phi(x, y), \tag{3.8}$$

for all $x, y \in X$. Then there exists a unique quartic mapping $Q : X \to Y$ satisfying equation (1.1) and

$$\|f(x) - Q(x)\|_Y \leq \frac{1}{2^{4\beta}\left|1 - L^i\right|}\Psi(x), \tag{3.9}$$

for all $x \in X$, where

$$\Psi(x) = \frac{K}{2^\beta}\left[\phi(x, 0) + \frac{(k^2-1)^\beta(7k^2+8)^\beta}{8^\beta}\phi(0, 0)\right].$$

Proof. Plugging (x, y) into $(0, 0)$ in (3.8), we get

$$\|f(0)\|_Y \leq \frac{2^\beta}{7^\beta k^{2\beta}(k^2-1)^\beta}\phi(0, 0). \tag{3.10}$$

Now, replacing (x, y) by $(x, 0)$ in (3.8) and eveness of f, we obtain

$$\left\|2[f(kx) - k^4 f(x) + \frac{(k^2-1)(7k^2+8)}{8}f(0)]\right\|_Y \leq \phi(x, 0), \tag{3.11}$$

for all $x \in X$. Using (3.10) in (3.11), one finds

$$\left\|f(kx) - k^4(x)\right\|_Y \leq \Psi(x), \tag{3.12}$$

for all $x \in X$. By Lemma 3.1, there exists a unique mapping $Q : X \to Y$ such that $Q(kx) = k^4 Q(x)$ and

$$\|f(x) - Q(x)\|_Y \leq \frac{1}{k^{4\beta}\left|1 - L^i\right|}\Psi(x),$$

for all $x \in X$. It remains to show that D is a quartic mapping. By (3.8), we have

$$\begin{aligned}\left\|\frac{1}{k^{4in}}D_k f\left(k^{in}x, k^{in}y\right)\right\|_Y &\leq k^{-4in\beta}\phi\left(k^{in}x, k^{in}y\right)\\ &\leq k^{-4in\beta}\left(k^{6i\beta}L\right)^n\phi(x, y)\\ &= L^n\phi(x, y),\end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. So $\|D_k Q(x, y)\|_Y = 0$ for all $x, y \in X$. Thus the mapping $U : X \to Y$ is quartic. It is not hard to prove the uniquess of D, which completes the proof. □

The forthcoming results are the applications of Theorem 3.5 to investigate stability of equation (1.1) in different versions devised by Th. M. Rassias and J. M. Rassias.

Corollary 3.6. *Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, and let Y be a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_Y$. Let c_3, a be positive numbers with $a \neq \frac{4\beta}{r}$ and $f : X \to Y$ be an even mapping with the condition $f(2x) = 2^4 f(x)$ satisfying*

$$\|D_k f(x, y)\|_Y \leq c_1\left(\|x\|_X^a + \|y\|_X^a\right),$$

for all $x, y \in X$. Then there exists a unique quartic mapping $Q : X \to Y$ satisfying the equation (1.1) and

$$\|f(x) - Q(x)\|_Y \leq \begin{cases}\frac{c_3 K}{2\beta(k^{4\beta} - k^{ar})}\|x\|_X^a, & a \in \left(0, \frac{4\beta}{r}\right)\\[2ex] \frac{c_3 K k^{ar}}{k^{4\beta}2^\beta(k^{ar} - k^{4\beta})}\|x\|_X^a, & a \in \left(\frac{4\beta}{r}, \infty\right),\end{cases}$$

for all $x \in X$, where

Proof. The proof follows directly by choosing $\phi(x,y) = c_3\left(\|x\|_X^a + \|y\|_X^a\right)$, for all $x, y \in X$ and $L = \frac{k^{ar}}{k^{4\beta}}$ in Theorem 3.5. $\square$

Corollary 3.7. *Let X be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_X$, and let Y be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_Y$. Let c_4, r, s be positive numbers with $a = r + s \neq \frac{4\beta}{r}$ and $f : X \to Y$ be an even mapping with the condition $f(2x) = 2^4 f(x)$ satisfying*

$$\|D_k f(x,y)\|_Y \leq c_2\left[\|x\|_X^r \|y\|_X^s + \left(\|x\|_X^{r+s} + \|y\|_X^{r+s}\right)\right],$$

for all non-zero $x, y \in X$. Then there exists a unique quartic mapping $Q : X \to Y$ satisfying the equation (1.1) and

$$\|f(x) - Q(x)\|_Y \leq \begin{cases} \frac{2c_4 K}{2^\beta(k^{4\beta} - k^{2ar})}\|x\|_X^a, & a \in \left(0, \frac{4\beta}{2r}\right) \\ \\ \frac{2c_4 K k^{2ar}}{k^{6\beta}2^\beta(k^{2ar} - k^{4\beta})}\|x\|_X^a, & a \in \left(\frac{4\beta}{2r}, \infty\right), \end{cases}$$

for all $x \in X$.

Proof. Considering $\phi(x,y) = c_4\left[\|x\|_X^r \|y\|_X^s + \left(\|x\|_X^{r+s} + \|y\|_X^{r+s}\right)\right]$, for all $x, y \in X$ and $L = \frac{k^{2ar}}{k^{4\beta}}$ in Theorem 3.5, we arrive at the desired results. $\square$

4. Counter-examples

In order to justify that the functional equation (1.1) is not stable for singular cases when $a = \frac{3\beta}{r}$ in Corollary 3.3 and $a = \frac{4\beta}{r}$ in Corollary 3.6, respectively, motivated from the renowned counter-example given by Z. Gajda [6], we illustrate counter-examples in this section.

Consider the function

$$\varphi(x) = \begin{cases} x^3, & \text{for } |x| < 1 \\ \\ 1, & \text{for } |x| \geq 1, \end{cases} \tag{4.1}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \sum_{n=0}^{\infty} 2^{-3n}\varphi(2^n x), \tag{4.2}$$

for all $x \in \mathbb{R}$. The function f serves as a counter-example for the fact that the functional equation (1.1) is not stable for $a = \frac{3\beta}{r}$ in Corollary 3.3 in the following theorem.

Theorem 4.1. *If the function f defined in (4.2) satisfies the functional inequality*

$$|D_k f(x,y)| \leq \frac{8^3\delta}{7}\left(|x|^3 + |y|^3\right), \tag{4.3}$$

where $\delta = \frac{9k^4+7k^2}{2} > 0$, for all $x, y \in \mathbb{R}$, then there do not exist a cubic mapping $C : \mathbb{R} \to \mathbb{R}$ and a constant $\epsilon > 0$ such that

$$|f(x) - C(x)| \leq \epsilon\,|x|^3, \qquad \text{for all } x \in \mathbb{R}.$$

Proof. First, let us show that f satisfies (4.3).

$$|f(x)| = \left|\sum_{n=0}^{\infty} 2^{-3n}\varphi(2^n x)\right| \leq \sum_{n=0}^{\infty} \frac{1}{2^{3n}} = \frac{8}{7}.$$

Therefore, f is bounded by $\frac{8}{7}$ on $\mathbb{R}$. If $|x|^3 + |y|^3 = 0$ or $|x|^3 + |y|^3 \geq \frac{1}{2}$, then

$$|D_k f(x,y)| \leq \frac{8\delta}{7} \leq \frac{8^2\delta}{7}\left(|x|^3 + |y|^3\right).$$

Suppose that $0 < |x|^3 + |y|^3 < \frac{1}{2}$. Then there exists a non-negative integer k such that

$$\frac{1}{2^{k+1}} \leq |x|^3 + |y|^3 < \frac{1}{2^k}. \tag{4.4}$$

Hence $2^k|x|^3 < 1$, $2^k|y|^3 < 1$ and $2^n(kx+y)$, $2^n(x+ky)$, $2^n(x-ky)$, $2^n(y-kx)$,$2^n(x+y)$, $2^n(x-y)$, $2^n(y-x)$ $2^n x$, $2^n y \in (-1,1)$ for all $n = 0,1,2,\ldots,k-1$. Hence for $n = 0,1,2,\ldots,k-1$,

$$\begin{aligned}
\varphi(2^n(x+ky)) &+ \varphi(2^n(kx+y)) + \varphi(2^n(x-ky)) + \varphi(2^n(y-kx)) \\
&- k^2\big[2\varphi(2^n(x+y)) + \varphi(2^n(x-y)) + \varphi(2^n(y-x))\big] \\
&- \frac{1}{4}k^2(k^2-1)\big[\varphi(2^n(2x)) + \varphi(2^n(2y))\big] \\
&+ 2(k^4-1)\big[\varphi(2^n(x)) + \varphi(2^n(y))\big] = 0.
\end{aligned} \tag{4.5}$$

From the definition of f and the inequality (4.4), one obtains that

$$\begin{aligned}
&|D_s f(x,y)| \\
&= \Big| \sum_{n=0}^{\infty} 2^{-3n}\varphi(2^n(x+ky)) \sum_{n=0}^{\infty} 2^{-3n}\varphi(2^n(kx+y)) \\
&\quad + \sum_{n=0}^{\infty} 2^{-3n}\varphi(2^n(x-ky)) + \sum_{n=0}^{\infty} 2^{-3n}\varphi(2^n(y-kx)) \\
&\quad - \sum_{n=0}^{\infty} 2^{-3n}k^2\big[2\varphi(2^n(x+y)) + \sum_{n=0}^{\infty} 2^{-3n}\varphi(2^n(x-y)) \\
&\quad + \sum_{n=0}^{\infty} 2^{-3n}\varphi(2^n(y-x))\big] - \sum_{n=0}^{\infty} 2^{-3n}\frac{1}{4}k^2(k^2-1)\big[\varphi(2^n(2x)) \\
&\quad + \sum_{n=0}^{\infty} 2^{-3n}\varphi(2^n(2y))\big] + 2(k^4-1)\big[\sum_{n=0}^{\infty} 2^{-3n}\varphi(2^n(x)) + \sum_{n=0}^{\infty} 2^{-3n}\varphi(2^n(y))\big]\Big| \\
&\leq \sum_{n=0}^{\infty} 2^{-3n}\Big|\varphi(2^n(x+ky)) + \varphi(2^n(kx+y)) + \varphi(2^n(x-ky)) \\
&\quad + \varphi(2^n(y-kx)) - k^2\big[2\varphi(2^n(x+y)) + \varphi(2^n(x-y)) + \varphi(2^n(y-x))\big] \\
&\quad - \frac{1}{4}k^2(k^2-1)\big[\varphi(2^n(2x)) + \varphi(2^n(2y))\big] + 2(k^4-1)\big[\varphi(2^n(x)) + \varphi(2^n(y))\big]\Big| \\
&\leq \sum_{n=0}^{\infty} 2^{-3n}\delta = \frac{2^{3(1-k)}\delta}{7} \leq \frac{8^3\delta}{7}\left(|x|^3 + |y|^3\right).
\end{aligned} \tag{4.6}$$

Therefore, f satisfies (4.3) for all $x, y \in \mathbb{R}$. Now, claim that the functional equation (1.1) is not stable for $a = \frac{3\beta}{r}$ in Corollary 3.3. Suppose on the contrary that there exists a cubic mapping $C : \mathbb{R} \to \mathbb{R}$ and a constant $\epsilon > 0$ such that

$$|f(x) - C(x)| \leq \epsilon |x|^3, \quad \text{for all } x \in \mathbb{R}.$$

Then there exists a constant $c \in \mathbb{R}$ such that $C(x) = cx^3$ for all rational numbers x (see [9]). So one finds that

$$|f(x)| \leq (\epsilon + |c|)\,|x|^3, \tag{4.7}$$

for all $x \in \mathbb{Q}$. Let $m \in \mathbb{N}$ with $m+1 > \epsilon + |c|$. If x is a rational number in $(0, 2^{-m})$, then $2^n x \in (0,1)$ for all $n = 0, 1, 2, \dots, m$, and for this x, one can have

$$f(x) = \sum_{n=0}^{\infty} 2^{-3n} \varphi(2^n x) \geq \sum_{n=0}^{m} 2^{-3n} \left(2^n x\right)^3 = (m+1)x^3 > (\epsilon + |c|)x^3, \tag{4.8}$$

which contradicts (4.7). Hence the functional equation (1.1) is not stable for $a = \frac{3\beta}{r}$ in Corollary 3.3. □

Consider again the function

$$\phi(x) = \begin{cases} x^4, & \text{for } |x| < 1 \\ 1, & \text{for } |x| \geq 1, \end{cases} \tag{4.9}$$

where $\phi : \mathbb{R} \to \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \sum_{n=0}^{\infty} 2^{-4n} \phi(2^n x), \tag{4.10}$$

for all $x \in \mathbb{R}$. The function f serves as a counter-example for the fact that the functional equation (1.1) is not stable for $a = \frac{4\beta}{r}$ in Corollary 3.6 in the following theorem. The proof is similar to Theorem 4.1, but for the comprehensiveness, we give below the entire proof of the theorem.

Theorem 4.2. *If the function f defined in (4.10) satisfies the functional inequality*

$$|D_k f(x,y)| \leq \frac{16^3 \delta}{15} \left(|x|^4 + |y|^4 \right), \tag{4.11}$$

where $\delta = \frac{9k^4+7k^2}{2} > 0$, for all $x, y \in \mathbb{R}$, then there do not exist a quartic mapping $Q : \mathbb{R} \to \mathbb{R}$ and a constant $\epsilon > 0$ such that

$$|f(x) - Q(x)| \leq \epsilon \, |x|^4, \qquad \text{for all } x \in \mathbb{R}.$$

Proof. First, let us show that f satisfies (4.11).

$$|f(x)| = \left| \sum_{n=0}^{\infty} 2^{-4n} \phi(2^n x) \right| \leq \sum_{n=0}^{\infty} \frac{1}{2^{4n}} = \frac{16}{15}.$$

Therefore, f is bounded by $\frac{16}{15}$ on $\mathbb{R}$. If $|x|^4 + |y|^4 = 0$ or $|x|^4 + |y|^4 \geq \frac{1}{2}$, then

$$|D_k f(x,y)| \leq \frac{16\delta}{15} \leq \frac{16^2 \delta}{15} \left(|x|^4 + |y|^4 \right).$$

Suppose that $0 < |x|^4 + |y|^4 < \frac{1}{2}$. Then there exists a non-negative integer k such that

$$\frac{1}{2^{k+1}} \leq |x|^4 + |y|^4 < \frac{1}{2^k}. \tag{4.12}$$

Hence $2^k |x|^4 < 1$, $2^k |y|^4 < 1$ and $2^n(kx+y)$, $2^n(x+ky)$, $2^n(x-ky)$, $2^n(y-kx)$, $2^n(x+y)$, $2^n(x-y)$, $2^n(y-x)$ $2^n x$, $2^n y \in (-1,1)$ for all $n = 0, 1, 2, \dots, k-1$. Hence for $n = 0, 1, 2, \dots, k-1$,

$$\begin{aligned} &\phi\left(2^n(x+ky)\right) + \phi\left(2^n(kx+y)\right) + \phi\left(2^n(x-ky)\right) + \phi\left(2^n(y-kx)\right) \\ &- k^2 \left[2\phi\left(2^n(x+y)\right) + \phi\left(2^n(x-y)\right) + \phi\left(2^n(y-x)\right) \right] \\ &- \frac{1}{4} k^2 (k^2-1) \left[\phi\left(2^n(2x)\right) + \phi\left(2^n(2y)\right) \right] \\ &+ 2(k^4-1) \left[\phi\left(2^n(x)\right) + \phi\left(2^n(y)\right) \right] = 0. \end{aligned} \tag{4.13}$$

From the definition of f and the inequality (4.12), one obtains that

$$\begin{aligned}
|D_k f(x,y)| = \Big| & \sum_{n=0}^{\infty} 2^{-4n}\phi\left(2^n(x+ky)\right) \sum_{n=0}^{\infty} 2^{-4n}\phi\left(2^n(kx+y)\right) \\
& + \sum_{n=0}^{\infty} 2^{-4n}\phi\left(2^n(x-ky)\right) + \sum_{n=0}^{\infty} 2^{-4n}\phi\left(2^n(y-kx)\right) \\
& - \sum_{n=0}^{\infty} 2^{-4n}k^2\big[2\phi\left(2^n(x+y)\right) + \sum_{n=0}^{\infty} 2^{-4n}\phi\left(2^n(x-y)\right) \\
& + \sum_{n=0}^{\infty} 2^{-4n}\phi\left(2^n(y-x)\right)\big] - \sum_{n=0}^{\infty} 2^{-4n}\frac{1}{4}k^2(k^2-1)\big[\phi\left(2^n(2x)\right) \\
& + \sum_{n=0}^{\infty} 2^{-4n}\phi\left(2^n(2y)\right)\big] + 2(k^4-1)\big[\sum_{n=0}^{\infty} 2^{-4n}\phi\left(2^n(x)\right) \\
& + \sum_{n=0}^{\infty} 2^{-4n}\phi\left(2^n(y)\right)\big]\Big| \\
\le & \sum_{n=0}^{\infty} 2^{-4n}\Big|\phi\left(2^n(x+ky)\right) + \phi\left(2^n(kx+y)\right) + \phi\left(2^n(x-ky)\right) \\
& + \phi\left(2^n(y-kx)\right) - k^2\big[2\phi\left(2^n(x+y)\right) + \phi\left(2^n(x-y)\right) + \phi\left(2^n(y-x)\right)\big] \\
& - \frac{1}{4}k^2(k^2-1)\big[\phi\left(2^n(2x)\right) + \phi\left(2^n(2y)\right)\big] \\
& + 2(k^4-1)\big[\phi\left(2^n(x)\right) + \phi\left(2^n(y)\right)\big]\Big| \\
\le & \sum_{n=0}^{\infty} 2^{-4n}\delta = \frac{2^{4(1-k)}\delta}{15} \le \frac{16^3\delta}{15}\left(|x|^4 + |y|^4\right).
\end{aligned} \tag{4.14}$$

Therefore, f satisfies (4.11) for all $x, y \in \mathbb{R}$. Now, claim that the functional equation (1.1) is not stable for $a = \frac{4\beta}{r}$ in Corollary 3.6. Suppose on the contrary that there exists a quartic mapping $Q : \mathbb{R} \to \mathbb{R}$ and a constant $\epsilon > 0$ such that

$$|f(x) - Q(x)| \le \epsilon\,|x|^4\,, \quad \text{for all } x \in \mathbb{R}.$$

Then there exists a constant $c \in \mathbb{R}$ such that $Q(x) = cx^4$ for all rational numbers x (see [9]). So one finds that

$$|f(x)| \le (\epsilon + |c|)\,|x|^4\,, \tag{4.15}$$

for all $x \in \mathbb{Q}$. Let $m \in \mathbb{N}$ with $m + 1 > \epsilon + |c|$. If x is a rational number in $(0, 2^{-m})$, then $2^n x \in (0, 1)$ for all $n = 0, 1, 2, \ldots, m$, and for this x, one can have

$$f(x) = \sum_{n=0}^{\infty} 2^{-4n}\phi(2^n x) \ge \sum_{n=0}^{m} 2^{-4n}\left(2^n x\right)^4 = (m+1)x^4 > (\epsilon + |c|)x^4, \tag{4.16}$$

which contradicts (4.15). Hence the functional equation (1.1) is not stable for $a = \frac{4\beta}{r}$ in Corollary 3.6. □

Acknowledgements

The authors are thankful to the anonymous reviewers for their worth and fruitful comments and suggestions to improve the quality of the paper.

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2** (1950), 64–66. 1

[2] L. Cadariu, V. Radu, *Fixed points and stability for functional equations in probabilistic metric and random normed spaces*, Fixed Point Theory and Appl., **2009** (2009), 18 pages. 1

[3] J. K. Chung, P. K. Sahoo, *On the general solution of a quartic functional equation*, Bull. Korean Math. Soc., **40** (2003), 565–576. 1

[4] D. Ž. Djoković, *A representation theorem for* $(X_1 - 1)(X_2 - 1)\dots(X_n - 1)$ *and its applications*, Ann. Polon. Math., **22** (1969/1970), 189–198. 1

[5] M. Eshaghi Gordji, A. Ebadian, S. Zolfaghari, *Stability of a functional equation deriving from cubic and quartic functions*, Abstr. Appl. Anal., **2008** (2008), 17 pages. 1

[6] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. & Math. Sci., **14** (1991), 431–434. 4

[7] P. Găvruţă, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., **184** (1994), 431–436. 1

[8] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27** (1941), 222–224. 1

[9] K. W. Jun, H. M. Kim, *On the stability of Euler-Lagrange type cubic mappings in quasi-Banach spaces*, J. Math. Anal. Appl. **332** (2007), 1335-1350. 4, 4

[10] T. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297–300. 1

[11] J. M. Rassias, *On approximation of approximately linear mappings by linear mappings*, J. Funct. Anal., **46** (1982), 126–130. 1

[12] J. M. Rassias, *On approximation of approximately linear mappings by linear mappings*, Bull. Sci. Math., **108** (1984), 445–446. 1

[13] J. M. Rassias, *Solution of problem of Ulam*, J. Approx. Theory, **57(3)** (1989), 268–273. 1

[14] J. M. Rassias, *Solution of the Ulam stability problem for quartic mappings*, Glas. Mat. Ser. III, **34** (1999), 243–252. 1

[15] J. M. Rassias, *Solution of the Ulam stability problem for cubic mappings*, Glas. Mat. Ser. III, **36** (2001), 63–72. 1

[16] J. M. Rassias, K. Ravi, B. V. Senthil Kumar, *A fixed point approach to Ulam-Hyers stability of duodecic functional equation in quasi-β-normed spaces*, Tbilisi Math. J., **10** (2017), 83–101. 1

[17] K. Ravi, M. Arunkumar, J. M. Rassias, *Ulam stability for the orthogonally general Euler-Lagrange type functional equation*, Int. J. Math. Stat., **3** (2008), 36–46. 1

[18] K. Ravi, J. M. Rassias, B. V. Senthil Kumar, *Ulam-Hyers stability of undecic functional equation in quasi-β-normed spaces: Fixed point method*, Tbilisi Math. J., **9** (2016), 83–103. 1

[19] P. K. Sahoo, *On a functional equation characterizing polynomials of degree three*, Bull. Inst. Math. Acad. Sinica, **32** (2004), 35–44. 1

[20] P. K. Sahoo, *A generalized cubic functional equation*, Acta Math. Sin. (Engl. Ser.), **21** (2005), 1159–1166. 1

[21] S. M. Ulam, *Problems in modern mathematics*, Science Editions John Wiley & Sons, Inc., New York, (1964). 1

[22] T. Z. Xu, J. M. Rassias, M. J. Rassias, W. X. Xu, *A fixed point approach to the stability of quintic and sextic functional equations in quasi-β-normed spaces*, J. Inequal. Appl., **2010** (2010), 23 pages. 3.1

[23] T. Z. Xu, J. M. Rassias, W. X. Xu, *A generalized mixed additive-cubic functional equation*, J. Comput. Anal. Appl., **13** (2011), 1273–1282. 1

[24] T. Z. Xu, J. M. Rassias, W. X. Xu, *A generalized mixed quadratic-quartic functional equation*, Bull. Malays. Math. Sci. Soc., **35** (2012), 633–649. 1

[25] S. Zolfaghari, A. Ebadian, S. Ostadbashi, M. De La Sen, M. Eshaghi Gordji, *A fixed point approach to the Hyers-Ulam stability of an AQ functional equation in probabilistic modular spaces*, Int. J. Nonlinear Anal. Appl., **4** (2013), 89–101. 1