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Fixed point theorems satisfying $C_\beta^\psi-$ condition and application to boundary value problem

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Abstract

In this paper, C_β^ψ-condition is defined and the existence and uniqueness of fixed points using this condition are discussed on metric spaces as well as on partially ordered metric spaces. As an application, we apply our result on a first order periodic boundary value problem to find its solution.

Keywords: Fixed point, C_β^ψ-condition, ordered metric spaces, boundary value problem.
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1. Introduction

Fixed point theory in metric spaces is an important branch of mathematical analysis, which is closely related to the existence and uniqueness of solutions of differential equations and integral equations. Especially in the last ten years, lot of publications have been done in the field of fixed point theory which are directly related to initial or boundary value problems (see:[2, 5, 6, 13, 16, 17, 18]). These problems are not only restricted to ordinary and partial differential equations while they are also useful to solve also fractional differential equations. Now theses days, several authors studied the existence of fixed point for weak contraction and generalized contractions in the sense of partially ordered sets. The first result in this direction was given by Ran and Reurings [20]. Later, in 2005, Nieto and Lopez [17, 18] extended the result of Ran and Reurings [20] and proved some results for non-decreasing functions in partially ordered set. They also discussed the applications of the fixed point theorems to the problem of existence and uniqueness of solutions

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of first order boundary value problems. After these results, number of results have been investigated to find fixed point in partially ordered metric spaces (see: [1, 4, 7, 8, 9, 10, 11, 15, 19, 21]).
In 2007, Suzuki [22] introduced the weaker C- contractive condition and proved some fixed point theorems. The existence and uniqueness of fixed points of such maps have also been extensively studied; see [12, 23].

Definition 1.1 ([22])**.** A mapping f on a metric space (X, d) satisfies the $C-$ Condition if

$$\frac{1}{2}d(x, fx) \leq d(x, y) \quad \Longrightarrow \quad d(fx, fy) \leq d(x, y), \ \forall\ x, y \in X.$$

We begin with the following definition and lemmas which are useful to prove our result.

Definition 1.2 ([14])**.** Let Ψ denote the class of function $\psi : [0, \infty) \to [0, \infty)$ (called altering distance function), which satisfies the following conditions:

Ψ1. ψ is continuous and non-decreasing,

Ψ2. $\psi(t) = 0 \Leftrightarrow t = 0$.

Lemma 1.3 ([24])**.** *If ψ is an altering distance function and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with condition $\psi(t) > \phi(t)$ for all $t > 0$, then $\phi(0) = 0$.*

Lemma 1.4 ([3])**.** *Let (X, d) be a metric spaces. Let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence in X then there exists an $\epsilon > 0$ and sequences of positive integers (m_k) and (n_k) with $m_k > n_k > k$ such that

$$d(x_{m_k}, x_{n_k}) \geq \epsilon, \quad d(x_{m_k-1}, x_{n_k}) < \epsilon$$

and

L1. $\lim\limits_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) = \epsilon,$

L2. $\lim\limits_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon,$

L3. $\lim\limits_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \epsilon.$

In this paper, we first define a $C^{\psi}_{\beta}-$ condition and then prove the existence and uniqueness of fixed points for self map both on metric spaces and on partially ordered metric spaces. As an application, we discuses here the existence and uniqueness of solutions of a first order periodic boundary value problem under some restricted conditions.

2. Fixed points on complete metric spaces

We define $C^{\psi}_{\beta}-$ condition as follows:

Definition 2.1. A mapping f on a metric space (X, d) is said to satisfying $C^{\psi}_{\beta}-$ condition if

$$\frac{1}{2}d(x, fx) \leq d(x, y) \quad \Longrightarrow \quad \psi(d(fx, fy)) \leq \beta(d(x, y)), \tag{2.1}$$

for all $x, y \in X$, $\psi \in \Psi$ and $\beta : [0, \infty) \to [0, \infty)$ is a continuous function.

We first prove a fixed point theorem on complete metric spaces and later, in next section, we formulate this result on complete metric spaces endowed with partial order.

Theorem 2.2. *Let (X, d) be a complete metric space and let $f : X \to X$ be a map satisfying $C^{\psi}_{\beta}-$ condition. If $\psi \in \Psi$ and $\beta : [0, \infty) \to [0, \infty)$ is a continuous function with condition*

$$\psi(t) > \beta(t), \forall\ t > 0. \tag{2.2}$$

Then f has a unique fixed point.

Proof. Let $x_0 \in X$ and define the sequence $\{x_n\}$ as follows:

$$x_n = fx_{n-1}, \forall n \in N. \tag{2.3}$$

If $x_n = x_{n+1}$ for some $n \in N$, then x_n is the fixed point of f. So assume that $x_n \neq x_{n+1}$,for all $n \in N$. Substituting $x = x_n$ and $y = fx_n = x_{n+1}$ in (2.1), we get

$$\frac{1}{2}d(x_n, fx_n) = \frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) \implies \psi(d(fx_n, fx_{n+1})) = \psi(d(x_{n+1}, x_{n+2})) \leq \beta(d(x_n, x_{n+1})) \tag{2.4}$$

By using property of ψ and β

$$d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}). \tag{2.5}$$

Similarly, we get

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n). \tag{2.6}$$

Thus we get a non-increasing sequence of functions such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r \geq 0. \tag{2.7}$$

However taking $\lim\limits_{n\to\infty}$ on both side of (2.3), we get $\psi(r) \leq \beta(r)$, which is a contradiction to (2.2) Thus we have $r = 0$, and hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r = 0. \tag{2.8}$$

Next we prove that the sequence $\{x_n\}$ is a Cauchy sequence. To prove this we suppose that $\{x_n\}$ is not a Cauchy sequence. Then for any $\epsilon > 0$, there exist two sub-sequences of positive integers m_k and n_k such that $n_k > m_k > k$ for all $k \in N$,

$$d(x_{m_k}, x_{n_k}) > \epsilon \quad and \quad d(x_{m_k}, x_{n_{k-1}}) \leq \epsilon. \tag{2.9}$$

Also the convergence of sequence $\{d(x_n, x_{n+1})\}$ implies that, for this $\epsilon > 0$, there exists $N_0 \in N$ such that $d(x_n, x_{n+1}) < \epsilon$ for all $n \geq N_0$. Let $N_1 = \max\{m_i, N_0\}$. Then for all $m_k > n_k \geq N_1$, we have

$$d(x_{n_k}, x_{n_k+1}) < \epsilon \leq d(x_{n_k}, x_{m_k}), \tag{2.10}$$

where $m_k > n_k$ and hence

$$\frac{1}{2}d(x_{n_k}, x_{n_k+1}) \leq d(x_{n_k}, x_{m_k}). \tag{2.11}$$

Now from (2.1), on substituting $x = x_{n_k}$ and $y = x_{m_k}$, we get

$$\psi(d(fx_{n_k}, fx_{m_k})) = \psi(d(x_{n_k+1}, x_{m_k+1})) \leq \beta(d(x_{n_k}, x_{m_k})). \tag{2.12}$$

Using Lemma 1.4 and taking limit as $k \to \infty$ in (2.12), we get $\psi(\epsilon) \leq \beta(\epsilon)$, this is a contradiction to (2.2) and hence by Lemma 1.3, we get $\epsilon = 0$. This contradicts the assumption that $\epsilon > 0$. Therefore our assumption is wrong. Hence $\{x_n\}$ is Cauchy and by the completeness of X it converges to a limit, say $z \in X$.
Now we assume that there exist $n \in N$ such that

$$d(x_n, z) < \frac{1}{2} d(x_n, x_{n+1}),$$

and

$$d(x_{n+1}, z) < \frac{1}{2} d(x_{n+1}, x_{n+2}).$$

Then we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, z) + d(x_{n+1}, z) \\ &< \frac{1}{2}\Big[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\Big] \\ &\leq d(x_n, x_{n+1}), \end{aligned}$$

this is a contradiction. Hence we must have $d(x_n, z) \geq \frac{1}{2} d(x_n, x_{n+1})$ or $d(x_{n+1}, z) \geq \frac{1}{2} d(x_{n+1}, x_{n+2})$, for all $n \in N$. Thus for a sub-sequence $\{n_k\}$ of N, we have

$$\frac{1}{2} d(x_{n_k}, f x_{n_k}) = \frac{1}{2} d(x_{n_k}, x_{n_k+1}) \leq d(x_{n_k}, z), \forall k \in N,$$

which implies

$$\psi(d(f x_{n_k}, f z)) = \beta(d(x_{n_k}, z)). \tag{2.13}$$

Letting $k \to \infty$ in (2.13), we get

$$\psi(d(z, fz)) \leq 0.$$

This is possible only if $d(z, fz) = 0$. That is, $fz = z$.
Finally, we prove the uniqueness of fixed points. Suppose on contrary that $x \neq y$ and $fx = x$ and $fy = y$. Then

$$0 = \frac{1}{2} d(x, fx) \leq d(x, y),$$

which implies

$$\psi(d(fx, fy)) = \psi(d(x, y)) \leq \beta(d(x, y)), \tag{2.14}$$

i.e

$$\psi(d(x, y)) \leq \beta(d(x, y)). \tag{2.15}$$

From the condition of (2.2) and Lemma 1.3, we get $d(x, y) = 0$. This means that $x = y$. Thus the map f has unique fixed point. This completes the proof. □

3. Fixed points on metric spaces with partial order

In this section, we define a condition similar to the condition in Theorem 2.2 and prove a fixed point theorem in the framework of partially ordered metric spaces. In this section, we show that uniqueness of a fixed point requires an additional condition (3.1).

Theorem 3.1. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and let $f : X \to X$ be a nondecreasing map satisfying C_β^ψ-condition. Also, suppose $\psi \in \Psi$ and $\beta : [0, \infty) \to [0, \infty)$ is a continuous function satisfying condition* (2.2)*. Suppose again that*

$$for\ each\ x, y \in X,\ there\ exists\ z \in X\ which\ is\ comparable\ to\ x\ and\ y. \tag{3.1}$$

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$. Then f has a unique fixed point in X.

Proof. Let $x_0 \in X$ satisfy $x_0 \preceq fx_0$. We define a sequence $\{x_n\}$ as follows:

$$x_n = fx_{n-1}, \forall n \in N. \tag{3.2}$$

If $x_n = x_{n+1}$ for some $n \in N$, then x_n is the fixed point of f. So assume that $x_n \neq x_{n+1}$ for all $n \in N$. Since $x_0 \preceq fx_0 = x_1$ and f is nondecreasing, then obviously

$$x_0 \preceq x_1 \preceq x_2 \cdots \preceq x_n \cdots. \tag{3.3}$$

Substituting $x = x_n$ and $y = fx_n = x_{n+1}$ in (2.1), we get

$$\frac{1}{2}d(x_n, fx_n) = \frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) \implies \psi(d(fx_n, fx_{n+1})) = \psi(d(x_{n+1}, x_{n+2})) \leq \beta(d(x_n, x_{n+1})). \tag{3.4}$$

By using property of ψ and β

$$d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}). \tag{3.5}$$

Similarly we get

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n). \tag{3.6}$$

Thus we get a non-increasing sequence of functions such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r \geq 0. \tag{3.7}$$

However, by taking $\lim\limits_{n\to\infty}$ on both side of (3.4), we get $\psi(r) \leq \beta(r)$, which is a contradiction to (2.2). Thus we have $r = 0$, and hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r = 0. \tag{3.8}$$

For remaining proof one can follow the lines of the proof of Theorem 2.2 and so we get $fx = x$.
To prove the uniqueness, we assume that there are two different fixed points, x and y, that is $x \neq y$ and $x = fx$ and $y = fy$. We consider the following cases:

Case 1. Suppose that x and y are comparable. Without loss of generality assume that $x \preceq y$. Then

$$0 = \frac{1}{2}d(x, fx) \leq d(x, y),$$

which implies

$$\psi(d(fx, fy)) = \psi(d(x, y)) \leq \beta(d(x, y)), \tag{3.9}$$

Thus from (2.2) and Lemma 1.3, we get $d(x, y) = 0$, *i.e* $x = y$.

Case 2. Assume that x and y are not comparable then from (3.1), there exists some $z \in X$ satisfying $x \preceq z$ and $y \preceq z$. Define the sequence $\{z_n\}$ as $z_0 = z, \quad z_{n+1} = fz_n, \forall\, n \in N$.
Since f is nondecreasing and $x \preceq z$, we have

$$x = fx \preceq fz = z_1 \cdots \implies x = fx \preceq fz_{n+1} = z_n, \quad n \in N. \tag{3.10}$$

If $x = z_{n_0}$ for some $n_0 \in N$, then $x = fx \preceq fz_{n_0} = z_{n_0}$ and, hence $x = F^k z = Z_k$ for all $k \geq n_0$. Thus, the sequence $\{z_n\}$ converges to the fixed point x, that is, $\lim\limits_{n\to\infty} d(x, z_n) = 0$. Assume that $x \neq z_n$ for all $n \in N$. Then we have

$$d(x, z_n) > \frac{1}{2} d(x, fx) = 0, \forall n \in N,$$

which implies that the contractive condition

$$\psi(d(fx, fz_n)) \leq \beta(d(x, z_n)), \tag{3.11}$$

consequently

$$d(x, z_{n+1}) \leq d(x, z_n),$$

that is, the sequence $\{d(x, z_n)\}$ is positive and decreasing, therefore, sequence $\{d(x, z_n)\}$ is convergent. Let $\lim\limits_{n\to\infty} d(x, z_n) = r \geq 0$. Taking the limit as $n \to \infty$ in (3.11), we get $\psi(r) \leq \beta(r)$, this is a contradiction to (2.2) and hence from Lemma 1.3 we have $r = 0$.
Thus we deduce that $\lim_{n\to\infty} d(x, z_n) = 0$.
Similarly, we can obtain $\lim_{n\to\infty} d(y, z_n) = 0$. This implies that $x = y$, This completes the proof of Theorem 3.1.

□

4. Applications

As an application, problem of ordinary differential equation is given with periodic boundary conditions. However, the existence and uniqueness conditions obtained here are weaker than those in the previous studies.

4.1. Existence theorem for solution of ordinary differential equations

In this section, we study the existence of solution for the following first -order periodic problem. The following example is inspired by [18, 24].
Consider the first order periodic boundary value problem:

$$\begin{cases} \frac{du}{dt} = f(t, u(t)), \quad t \in [0, T], \\ u(0) = u(T), \end{cases} \tag{4.1}$$

where $T > 0$ and $f : I \times R \to R$ is continuous function. Consider the space $C(I)$ of continuous function defined in I. Clearly this space with the metric given by

$$d(u, v) = \sup\{|u(t) - v(t)| : t \in I\}; \text{ for all } u, v \in C(I),$$

is a complete metric space. C(I) can also be equipped with the partial order given by

$$u, v \in C(I), u \preceq v \Leftrightarrow u(t) \leq v(t) \quad \text{for all } t \in I.$$

Obviously, $(C(I), \preceq)$ satisfies the condition (3.1). Indeed, it is obvious that for every pair $u(t), v(t) \in X$, we have $u(t) = \max\{u(t), v(t)\}$ and $v(t) = \max\{u(t), v(t)\}$.

Definition 4.1 ([24]). A lower solution of the problem (4.1) is a function $\alpha \in C^{(1)}(I)$ such that

$$\begin{cases} \alpha'(t) \leq f(t, \alpha(t)), & t \in [0, T], \\ \alpha(0) \leq \alpha(T). \end{cases}$$

Theorem 4.2. *Consider the problem ((4.1)) with $f : I \times R \to R$ is continuous, and suppose that there exists $\lambda, \alpha > 0$, with $0 < \alpha < \lambda$, such that for all $u, v \in C[0, T]$ satisfying $u \leq v$, the following condition holds:*

$$\begin{cases} v'(t) \geq f(t, u(t)), & t \in [0, T], \\ v(0) \geq v(T), \end{cases} \tag{4.2}$$

and

$$\alpha \leq \Big(\frac{2\lambda(e^{\lambda t} - 1)}{T(e^{\lambda t} + 1)}\Big)^{\frac{1}{2}}, \tag{4.3}$$

such that for all $u, v \in R$ with $u \geq v$, we have

$$0 \leq (f(t, u(t)) - f(t, v(t))) + \lambda(u(t) - v(t)) \leq \alpha\sqrt{(u(t) - v(t)) \cdot log[(u(t) - v(t))^2 + 1]}. \tag{4.4}$$

Then the existence of a lower solution for (4.1) provides the existence of a unique solution of (4.1).

Proof. We can write (4.1) as:

$$\begin{cases} \frac{du}{dt} + \lambda u(t) = f(t, u(t)) + \lambda u(t), & t \in [0, T], \\ u(0) = u(T). \end{cases} \tag{4.5}$$

This problem is equivalent to the integral equation

$$u(t) = \int_0^T G(t, s)[f(s, u(s)) + \lambda u(s)]ds,$$

where $G(t, s)$ is the green function given by

$$G(t, s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & if \quad 0 \leq s < t \leq T \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & if \quad 0 \leq t < s \leq T. \end{cases}$$

Define $F : C(I) \to C(I)$ by

$$(Fu)(t) = \int_0^T G(t, s)[f(s, u(s)) + \lambda u(s)]ds.$$

Now, if $u \in C(I)$ is a fixed point of F, then $u \in C^1(I)$ is a solution of (3.9).
Next we claim that the hypotheses in Theorem 3.1 are satisfied.
Assume that $u \leq v$ are functions in $C(I)$ satisfying (4.2), therefore we can rewrite the inequality $v'(t) \geq f(t, u(t))$ as

$$v'(t) + \lambda v(t) \geq f(t, u(t)) + \lambda u(t).$$

Further we can write it as (follow from [13])

$$v(t) \geq \int_0^T G(t, s)[f(s, u(s)) + \lambda u(s)]ds.$$

This implies

$$\sup\{|v(t)-u(t)| : t\in I\} \geq \sup\{|Fu(t)-u(t)| : t\in I\},$$

or

$$d(u,v) \geq d(Fu,v) \geq \frac{1}{2}d(Fu,u).$$

Moreover, Since $G(t,s)>0$ for $t\in I$, we have

$$\begin{aligned}(Fu)(t) &= \int_0^T G(t,s)[f(s,u(s))+\lambda u(s)]ds\\ &\leq \int_0^T G(t,s)[f(s,v(s))+\lambda v(s)]ds = (Fv)(t).\end{aligned}$$

Thus F is nondecreasing. Also for all $u\leq v$, we have

$$\begin{aligned}d(Fu,Fv) &= \sup_{t\in I}|(Fu)(t)-(Fv)(t)| = \sup_{t\in I}((Fu)(t)-(Fv)(t))\\ &= \sup_{t\in I}\int_0^T G(t,s)[f(s,u(s))+\lambda u(s)-f(s,v(s))-\lambda v(s)]ds\\ &= \sup_{t\in I}\int_0^T G(t,s)\alpha\sqrt{(u(s)-v(s))\cdot\log[(u(s)-v(s))^2+1]}ds. \qquad (4.6)\end{aligned}$$

Using Cauchy-Schwarz inequality in the last integral, we get

$$\begin{aligned}&\int_0^T G(t,s)\alpha\sqrt{(u(s)-v(s))\cdot\log[(u(s)-v(s))^2+1]}ds\\ &\leq \Big(\int_0^T G(t,s)^2ds\Big)^{\frac{1}{2}}\Big(\int_0^T \alpha^2(u(s)-v(s))\cdot\log[(u(s)-v(s))^2+1]ds\Big)^{\frac{1}{2}}. \qquad (4.7)\end{aligned}$$

Also, Yan *et al.* [24] proved that

$$\int_0^T G(t,s)^2ds = \frac{e^{\lambda t}+1}{2\lambda(e^{\lambda t}-1)}. \qquad (4.8)$$

Now we consider the second integral in (4.7), we get

$$\int_0^T \alpha^2(u(s)-v(s))\log[(u(s)-v(s))^2+1]ds \leq \alpha^2 d(u,v)\cdot\log[d(u,v)^2+1]\cdot T. \qquad (4.9)$$

Using (4.7), (4.8) and (4.9) in (4.6), we get

$$\begin{aligned}d(Fu,Fv) &\leq \sup_{t\in I}\Big(\frac{e^{\lambda t}+1}{2\lambda(e^{\lambda t}-1)}\Big)^{\frac{1}{2}}.\Big(\alpha^2 d(u,v)\cdot\log[d(u,v)^2+1]\cdot T\Big)^{\frac{1}{2}}. \qquad (4.10)\\ &\leq \sup_{t\in I}\Big(\frac{e^{\lambda t}+1}{2\lambda(e^{\lambda t}-1)}\Big)^{\frac{1}{2}}\Big(\Big(\frac{2\lambda(e^{\lambda t}-1)}{T(e^{\lambda t}+1)}\Big)d(u,v)\cdot\log[d(u,v)^2+1]\cdot T\Big)^{\frac{1}{2}}. \qquad (4.11)\end{aligned}$$

Consequently, we get

$$d(Fu,Fv)^2 \leq d(u,v)\cdot\log[d(u,v)^2+1]. \qquad (4.12)$$

Assuming $\psi(t)=t^2$, and $\beta(t)=t\log[t^2+1]$. Clearly, $\psi\in\Psi$ and $\psi(t)$ and $\beta(t)$ satisfy the condition $\psi(t)>\beta(t)$ for all $t>0, t\in I$.
Hence from (4.12), we obtain

$$\psi(d(Fu,Fv)) \leq \beta(d(u,v)).$$

Also, Yan *et al.* [24] proved that $\alpha(t)\leq F(\alpha(t))$, be a lower solution for (3.9). Thus we can say that all the conditions of Theorem 3.1 are satisfied and hence F has a unique fixed point. Consequently, $u\in C^1(I)$ is a solution of (4.1). □

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