Available online at www.vonneumann-publishing.com/fatma
Func. Anal.-TMA 3 (2017), 35–43
Research Article

Functional Analysis: Theory, Methods & Applications
Journal Homepage: http://vonneumann-publishing.com/fatma

Existence results for positive solutions of Riemann–Liouville fractional order three-point boundary value problems

Boddu Muralee Bala Krushna

Department of Mathematics, MVGR College of Engineering (Autonmous), Vizianagaram, 535 005, India.

Communicated by R. Saadati

Abstract

This paper deals with the results of an investigation on sufficient conditions for the existence of at least three positive solutions to a fractional order three-point boundary value problems with Riemann–Liouville type by means of fixed point theorem on a cone in a Banach space.

Keywords: Fractional derivative, boundary value problem, Green's function, positive solution.
2010 MSC: 26A33, 34B15, 34B18.

1. Introduction

This paper is concerned with the existence of multiple positive solutions to the fractional order differential equations

$$D_{0^+}^{r_1}x(t) + f\big(t, x(t)\big) = 0,\ t \in (0,1), \tag{1.1}$$

satisfying three-point boundary conditions

$$\left.\begin{aligned} x^{(k)}(0) = 0,\ k = 0, 1, \cdots, n-2,\\ \beta D_{0^+}^{r_2}x(1) = \alpha D_{0^+}^{r_2}x(\xi),\end{aligned}\right\} \tag{1.2}$$

where $r_1 \in (n-1, n]$, $n \geq 2$, $\xi \in (0,1)$, $r_2 \in (1, r_1)$, α, β are positive constants and $D_{0^+}^{r_1}, D_{0^+}^{r_2}$ are the standard Riemann–Liouville fractional order derivatives.

*Corresponding author
Email address: muraleebalu@yahoo.com (Boddu Muralee Bala Krushna)

Received 2015-08-06

Fractional calculus is the field of mathematical analysis which unifies the theories of integration and differentiation of any arbitrary real order. In describing the properties of various real materials, the derivatives and integrals of non-integer order are very much suitable. They arise in many engineering and scientific disciplines like mathematical modeling of systems and processes in various fields such as physics, mechanics, control systems, flow in porous media, electromagnetics and viscoelasticity.

Boundary value problems associated with linear as well as nonlinear ordinary or fractional order differential equations have achieved a great deal of interest and play a pivotal role in many areas of applied mathematics like engineering design and manufacturing. Major established industries such as automobile, chemical, electronics and communications, biotechnology and nanotechnology rely on boundary value problems to simulate complex phenomena at various scales for designing and manufacturing of high technological products and in these applied settings, positive solutions are meaningful.

There has been much interest created in establishing solutions, positive solutions and multiple positive solutions for fractional order boundary value problems. See for example, the papers of Bai and Lü [3], Kauffman and Mboumi [6], Benchohra, Henderson, Ntoyuas and Ouahab [4], Ahmed and Nieto [2], Prasad and Krushna [11, 12, 14], Prasad, Krushna and Sreedhar [13].

The rest of the paper is organized as follows. In Section 2, the Green's function for the associated linear fractional order boundary value problem is constructed and the bounds for the Green's function are estimated. In Section 3, sufficient conditions for the existence of at least three positive solutions to the fractional order boundary value problem (1.1)-(1.2) are established by using Leggett–Williams fixed point theorem. In Section 4, as an application, the results are demonstrated with an example.

2. Green's function and bounds

In this section, the Green's function for the associated linear fractional order boundary value problem is constructed and the bounds for the Green's function are estimated, which are essential to establish the main results.

Lemma 2.1. *Let $\Delta = \mathcal{K}\Gamma(r_1) \neq 0$. If $h(t) \in C[0,1]$, then the fractional order differential equations*

$$D_{0^+}^{r_1} x(t) + h(t) = 0, \ t \in (0,1), \tag{2.1}$$

satisfying the boundary conditions (1.2)*, has a unique solution*

$$x(t) = \int_0^1 H(t,s)h(s)ds,$$

where $H(t,s)$ is the Green's function for the problem (2.1), (1.2) *and is given by*

$$H(t,s) = \begin{cases} \underset{t\in[0,\xi]}{H(t,s)} = \begin{cases} H_{11}(t,s), & 0 \le t \le s \le \xi < 1, \\ H_{12}(t,s), & 0 \le s \le \min\{t,\xi\} < 1, \end{cases} \\ \\ \underset{t\in[\xi,1]}{H(t,s)} = \begin{cases} H_{13}(t,s), & 0 \le \max\{t,\xi\} \le s \le 1, \\ H_{14}(t,s), & 0 < \xi \le s \le t \le 1, \end{cases} \end{cases} \tag{2.2}$$

$$\begin{aligned} H_{11}(t,s) =& \frac{\Big[\beta t^{r_1-1}(1-s)^{r_1-r_2-1} - \alpha t^{r_1-1}(\xi-s)^{r_1-r_2-1}\Big]}{\Delta}, \\ H_{12}(t,s) =& \frac{\Big[\beta t^{r_1-1}(1-s)^{r_1-r_2-1} - \mathcal{K}(t-s)^{r_1-1} - \alpha t^{r_1-1}(\xi-s)^{r_1-r_2-1}\Big]}{\Delta}, \\ H_{13}(t,s) =& \frac{\Big[\beta t^{r_1-1}(1-s)^{r_1-r_2-1}\Big]}{\Delta}, \\ H_{14}(t,s) =& \frac{\Big[\beta t^{r_1-1}(1-s)^{r_1-r_2-1} - \mathcal{K}(t-s)^{r_1-1}\Big]}{\Delta}, \\ \mathcal{K} =& \beta - \alpha\xi^{r_1-r_2-1}. \end{aligned}$$

Proof. Let $x(t) \in C^{r_1}[0,1]$ be the solution of fractional order boundary value problem given by (2.1) and (1.2). An equivalent integral equation for (2.1) is given by

$$x(t) = \frac{-1}{\Gamma(r_1)} \int_0^t (t-s)^{r_1-1} h(s)ds + c_1 t^{r_1-1} + \cdots + c_n t^{r_1-n}.$$

Utilizing the conditions (1.2), one can obtain $c_n = c_{n-1} = \cdots = c_2 = 0$ and

$$c_1 = \frac{1}{\Delta}\left[\beta \int_0^1 (1-s)^{r_1-r_2-1} h(s)ds - \alpha \int_0^\xi (\xi-s)^{r_1-r_2-1} h(s)ds\right].$$

Hence the unique solution of the problem given by (2.1) and (1.2) is

$$\begin{aligned} x(t) &= \frac{t^{r_1-1}}{\Delta}\Big[\beta \int_0^1 (1-s)^{r_1-r_2-1} h(s)ds - \alpha \int_0^\xi (\xi-s)^{r_1-r_2-1} h(s)ds\Big] \\ &\quad - \frac{\mathcal{K}}{\Delta}\int_0^t (t-s)^{r_1-r_2-1} h(s)ds \\ &= \int_0^1 H(t,s)h(s)ds. \end{aligned}$$

□

Lemma 2.2. *Let $\mathcal{K} > 0$ and $\tau \in (0,1)$. Then the Green's function $H(t,s)$ given in (2.2) satisfies the inequalities*

$$\begin{aligned} (i)\ & H(t,s) \geq 0,\ \forall\ (t,s) \in [0,1]\times[0,1], \\ (ii)\ & H(t,s) \leq H(1,s),\ \forall\ (t,s) \in [0,1]\times[0,1], \\ (iii)\ & H(t,s) \geq \tau^{r_1-1} H(1,s),\ \forall\ (t,s) \in [\tau,1]\times[0,1]. \end{aligned}$$

Proof. Consider the Green's function $H(t,s)$ given by (2.2).
Case 1. Let $0 \leq t \leq s \leq \xi \leq 1$. Then

$$\begin{aligned} H_{11}(t,s) &= \frac{\Big[\beta t^{r_1-1}(1-s)^{r_1-r_2-1} - \alpha t^{r_1-1}(\xi-s)^{r_1-r_2-1}\Big]}{\Delta} \\ &\geq \frac{\Big[\beta t^{r_1-1}(1-s)^{r_1-r_2-1} - \alpha t^{r_1-1}(\xi-\xi s)^{r_1-r_2-1}\Big]}{\Delta} \\ &= \frac{t^{r_1-1}\Big[\mathcal{K}\Big(1 + r_2 s + O(s^2)\Big)\Big](1-s)^{r_1-1}}{\Delta} \geq 0. \end{aligned}$$

Case 2. Let $0 \leq s \leq \min\{t,\xi\} \leq 1$. Then

$$\begin{aligned} H_{12}(t,s) &= \frac{\Big[\beta t^{r_1-1}(1-s)^{r_1-r_2-1} - \mathcal{K}(t-s)^{r_1-1} - \alpha t^{r_1-1}(\xi-s)^{r_1-r_2-1}\Big]}{\Delta} \\ &\geq \frac{\Big[\beta t^{r_1-1}(1-s)^{r_1-r_2-1} - \mathcal{K}(t-ts)^{r_1-1} - \alpha t^{r_1-1}(\xi-\xi s)^{r_1-r_2-1}\Big]}{\Delta} \\ &= \frac{t^{r_1-1}\Big[\mathcal{K}\Big((1-s)^{-r_2} - 1\Big)\Big](1-s)^{r_1-1}}{\Delta} \\ &= \frac{t^{r_1-1}\Big[r_2 s\mathcal{K} + O(s^2)\Big](1-s)^{r_1-1}}{\Delta} \geq 0. \end{aligned}$$

Case 3. Let $0 \le \max\{t,\xi\} \le s \le 1$. Then

$$H_{13}(t,s) = \frac{\Big[\beta t^{r_1-1}(1-s)^{r_1-r_2-1}\Big]}{\Delta} \ge 0.$$

Case 4. Let $0 \le \xi \le s \le t \le 1$. Then

$$\begin{aligned} H_{14}(t,s) &= \frac{\Big[\beta t^{r_1-1}(1-s)^{r_1-r_2-1} - \mathcal{K}(t-s)^{r_1-1}\Big]}{\Delta} \\ &\ge \frac{\Big[\beta t^{r_1-1}(1-s)^{r_1-r_2-1} - \mathcal{K}(t-ts)^{r_1-1}\Big]}{\Delta} \\ &= \frac{t^{r_1-1}\Big[\beta r_2 s + \alpha\xi^{r_1-r_2-1} + O(s^2)\Big](1-s)^{r_1-1}}{\Delta} \ge 0. \end{aligned}$$

Now we establish the inequality (ii).

Case (i). Let $0 \le t \le s \le \xi \le 1$. Then, we have

$$\begin{aligned} \frac{\partial H_{11}(t,s)}{\partial t} &= \frac{(r_1-1)\Big[\beta t^{r_1-2}(1-s)^{r_1-r_2-1} - \alpha t^{r_1-2}(\xi-s)^{r_1-r_2-1}\Big]}{\Delta} \\ &\ge \frac{(r_1-1)\Big[\beta t^{r_1-2}(1-s)^{r_1-r_2-1} - \alpha t^{r_1-2}(\xi-\xi s)^{r_1-r_2-1}\Big]}{\Delta} \\ &= \frac{(r_1-1)t^{r_1-1}\Big[\mathcal{K}\Big(1 + r_2 s + O(s^2)\Big)\Big](1-s)^{r_1-1}}{\Delta} \ge 0. \end{aligned}$$

Therefore, $H_{11}(t,s)$ is increasing in t, which implies $H_{11}(t,s) \le H_{11}(1,s)$.

Case (ii). Let $0 \le s \le \min\{t,\xi\} \le 1$. Then, we have

$$\begin{aligned} &\frac{\partial H_{12}(t,s)}{\partial t} \\ &= \frac{(r_1-1)\Big[\beta t^{r_1-2}(1-s)^{r_1-r_2-1} - \mathcal{K}(t-s)^{r_1-2} - \alpha t^{r_1-2}(\xi-s)^{r_1-r_2-1}\Big]}{\Delta} \\ &\ge \frac{(r_1-1)\Big[\beta t^{r_1-2}(1-s)^{r_1-r_2-1} - \mathcal{K}(t-ts)^{r_1-2} - \alpha t^{r_1-2}(\xi-\xi s)^{r_1-r_2-1}\Big]}{\Delta} \\ &= \frac{(r_1-1)t^{r_1-2}\Big[\mathcal{K}\Big((1-s)^{-(r_2-1)} - 1\Big)\Big](1-s)^{r_1-2}}{\Delta} \\ &= \frac{(r_1-1)t^{r_1-2}\Big[(r_2-1)s\mathcal{K} + O(s^2)\Big](1-s)^{r_1-2}}{\Delta} \ge 0. \end{aligned}$$

Therefore, $H_{12}(t,s)$ is increasing in t, which implies $H_{12}(t,s) \le H_{12}(1,s)$.

Case (iii). Let $0 \le \max\{t,\xi\} \le s \le 1$. Then, we have

$$\frac{\partial H_{13}(t,s)}{\partial t} = \frac{(r_1-1)\Big[\beta t^{r_1-2}(1-s)^{r_1-r_2-1}\Big]}{\Delta} \ge 0.$$

Therefore, $H_{13}(t,s)$ is increasing in t, which implies $H_{13}(t,s) \le H_{13}(1,s)$.

Case (iv). Let $0 \le \xi \le s \le t \le 1$. Then, we have

$$\begin{aligned}\frac{\partial H_{14}(t,s)}{\partial t} &= \frac{(r_1-1)\Big[\beta t^{r_1-2}(1-s)^{r_1-r_2-1} - \mathcal{K}(t-s)^{r_1-2}\Big]}{\Delta}\\ &\ge \frac{(r_1-1)\Big[\beta t^{r_1-2}(1-s)^{r_1-r_2-1} - \mathcal{K}(t-ts)^{r_1-2}\Big]}{\Delta}\\ &= \frac{(r_1-1)t^{r_1-2}\Big[\beta(r_2-1)s + O(s^2) + \alpha\xi^{r_1-r_2-1}\Big](1-s)^{r_1-2}}{\Delta} \ge 0.\end{aligned}$$

Therefore, $H_{14}(t,s)$ is increasing in t, which implies $H_{14}(t,s) \le H_{14}(1,s)$.
Finally we can establish the inequality (iii).
Case (a). Let $0 \le t \le s \le \xi \le 1$ and $t \in [\tau, 1]$. Then

$$\begin{aligned}H_{11}(t,s) &= \frac{\Big[\beta t^{r_1-1}(1-s)^{r_1-r_2-1} - \alpha t^{r_1-1}(\xi-s)^{r_1-r_2-1}\Big]}{\Delta}\\ &= \frac{t^{r_1-1}\Big[\beta(1-s)^{r_1-r_2-1} - \alpha(\xi-s)^{r_1-r_2-1}\Big]}{\Delta}\\ &= t^{r_1-1}H_{11}(1,s) \ge \tau^{r_1-1}H_{11}(1,s).\end{aligned}$$

Case (b). Let $0 \le s \le \min\{t,\xi\} \le 1$ and $t \in [\tau, 1]$. Then

$$\begin{aligned}H_{12}(t,s) &= \frac{\Big[\beta t^{r_1-1}(1-s)^{r_1-r_2-1} - \mathcal{K}(t-s)^{r_1-1} - \alpha t^{r_1-1}(\xi-s)^{r_1-r_2-1}\Big]}{\Delta}\\ &\ge \frac{\Big[\beta t^{r_1-1}(1-s)^{r_1-r_2-1} - \mathcal{K}(t-ts)^{r_1-1} - \alpha t^{r_1-1}(\xi-s)^{r_1-r_2-1}\Big]}{\Delta}\\ &= \frac{t^{r_1-1}\Big[\beta(1-s)^{r_1-r_2-1} - \mathcal{K}(1-s)^{r_1-1} - \alpha(\xi-s)^{r_1-r_2-1}\Big]}{\Delta}\\ &= t^{r_1-1}H_{12}(1,s) \ge \tau^{r_1-1}H_{12}(1,s).\end{aligned}$$

Case (c). Let $0 \le \max\{t,\xi\} \le s \le 1$ and $t \in [\tau, 1]$. Then

$$\begin{aligned}H_{13}(t,s) &= \frac{\Big[\beta t^{r_1-1}(1-s)^{r_1-r_2-1}\Big]}{\Delta}\\ &= t^{r_1-1}H_{13}(1,s) \ge \tau^{r_1-1}H_{13}(1,s).\end{aligned}$$

Case (d). Let $0 \le \xi \le s \le t \le 1$ and $t \in [\tau, 1]$. Then

$$\begin{aligned}H_{14}(t,s) &= \frac{\Big[\beta t^{r_1-1}(1-s)^{r_1-r_2-1} - \mathcal{K}(t-s)^{r_1-1}\Big]}{\Delta}\\ &\ge \frac{t^{r_1-1}\Big[\beta(1-s)^{r_1-r_2-1} - \mathcal{K}(1-s)^{r_1-1}\Big]}{\Delta}\\ &= t^{r_1-1}H_{14}(1,s) \ge \tau^{r_1-1}H_{14}(1,s),\end{aligned}$$

where $\tau \in (0,1)$ satisfies $\displaystyle\int_\tau^1 H(1,s)ds > 0$. □

To establish the existence of positive solutions to the fractional order boundary value problem (1.1)-(1.2) by using the following Leggett–Williams fixed point theorem.

Theorem 2.3 ([8]). *Let $T : \overline{P}_c \to \overline{P}_c$ be completely continuous and S be a nonnegative continuous concave functional on P such that $S(y) \le \|y\|$ for all $y \in \overline{P}_c$. Suppose that there exist a, b, c and d with $0 < d < a < b \le c$ such that*

$$\begin{aligned} &(i)\ \Big\{y \in P(S,a,b) : S(y) > a\Big\} \ne \emptyset \text{ and } S(Ty) > a \text{ for } y \in P(S,a,b),\\ &(ii)\ \|Ty\| < d \text{ for } \|y\| \le d,\\ &(iii)\ S(Ty) > a \text{ for } y \in P(S,a,c) \text{ with } \|Ty\| > b. \end{aligned}$$

Then T has at least three fixed points y_1, y_2, y_3 in $\overline{P}_c$ satisfying

$$\|y_1\| < d, a < S(y_2), \|y_3\| > d,\ S(y_3) < a.$$

3. Main Results

In this section, sufficient conditions for the existence of at least three positive solutions to the fractional order three-point boundary value problems (1.1)-(1.2) is established by utilizing Leggett–Williams fixed point theorem.

Let a' and b' be be two real numbers such that $0 < a' < b'$ and S be a nonnegative continuous concave functional on a cone P.

Define the following convex sets

$$P_{a'} = \Big\{x \in P : \|x\| < a'\Big\} \text{ and}$$
$$P\big(S, a', b'\big) = \Big\{x \in P : a' \le S(x), \|x\| < b'\Big\}.$$

For $x \in P$, we have

$$S\big(x(t)\big) = \min_{t\in[\tau,1]} \Big\{x(t)\Big\}. \tag{3.1}$$

Consider the Banach space $E = \Big\{x : x \in C[0,1]\Big\}$ equipped with the norm

$$\|x\| = \max_{t\in[0,1]} |x(t)|.$$

Define a cone $P \subset E$ by

$$P = \Big\{x \in E : x(t) \ge 0,\ t \in [0,1] \text{ and } \min_{t\in[\tau,1]} x(t) \ge \eta\|x\|\Big\},$$

where $\eta = \tau^{r_1-1}$.

Let $T : P \to B$ be the operator defined by

$$Tx(t) = \int_0^1 H(t,s) f\big(s, x(s)\big) ds, \quad t \in [0,1]. \tag{3.2}$$

Let

$$\mathcal{M} = \max_{t\in[0,1]} \Big\{ \int_0^1 H(t,s)ds\Big\} \text{ and } \mathcal{N} = \min_{t\in[\tau,1]} \Big\{ \int_\tau^1 H(t,s)ds\Big\}.$$

Lemma 3.1. *The operator T defined by (3.2) is a self map on P.*

Proof. Let $x \in P$. Clearly, $Tx(t) \geq 0$ for $t \in [0,1]$. Also, for $x \in P$,

$$\|Tx\| \leq \int_0^1 H(1,s) f\big(s, x(s)\big) ds$$

and

$$\begin{aligned}
\min_{t\in[\tau,1]} Tx(t) &= \min_{t\in[\tau,1]} \int_0^1 H(t,s) f\big(s, x(s)\big) ds \\
&\geq \tau^{r_1-1} \int_0^1 H(1,s) f\big(s, x(s)\big) ds \\
&\geq \tau^{r_1-1} \|Tx\| = \eta \|Tx\|.
\end{aligned}$$

Hence $Tx \in P$ and so $T : P \to P$. Standard arguments involving the Arzela–Ascoli theorem shows that T is completely continuous. $\square$

Theorem 3.2. *Assume that there exist real numbers k_1, k_2 and c with $0 < k_1 < k_2 < \dfrac{k_2}{\eta} < k_3$ such that such that the following hold, such that f satisfies the following conditions:*

$$\begin{aligned}
&(H1)\ f\big(t, x\big) < \frac{k_1}{\mathcal{N}},\ \textit{for}\ t \in [0,1],\ x \in \Big[0, k_1\Big], \\
&(H2)\ f\big(t, x\big) > \frac{k_2}{\mathcal{M}},\ \textit{for}\ t \in [\tau,1],\ x \in \Big[k_2, \frac{k_2}{\eta}\Big], \\
&(H3)\ f\big(t, x\big) < \frac{k_3}{\mathcal{N}},\ \textit{for}\ t \in [0,1],\ x \in \Big[0, k_3\Big].
\end{aligned}$$

Then the fractional order boundary value problem (1.1)-(1.2) *has at least three positive solutions.*

Proof. We seek three fixed points $w_1, w_2, w_3 \in P$ of T defined by (3.2). It is easy to check that S is a nonnegative continuous concave functional on P with $S(x) \leq \|x\|$ for $x \in P$ and from Lemma 3.1, the operator T is completely continuous and fixed points of T are solutions of the fractional order boundary value problem (1.1)-(1.2). First we prove that if there exist a positive number r such that $f\big(t, x(t)\big) < \dfrac{r}{\mathcal{N}}$, for $t \in [0,1]$ and $x \in [0,r]$, then $T : \overline{P}_r \to \overline{P}_r$. For $x \in P_r$ and $t \in [0,1]$, we have

$$\begin{aligned}
\|Tx\| &= \max_{t\in[0,1]} \Big\{ \Big| \int_0^1 H(t,s) f\big(s, x(s)\big) ds \Big| \Big\} \\
&\leq \frac{r}{\mathcal{N}} \min_{t\in[\tau,1]} \int_\tau^1 H(t,s) ds = r.
\end{aligned}$$

Thus $\|Tx\| \leq r$. Hence $Tx \in P_r$. Hence, we have shown that if $(H1)$ and $(H3)$ hold then T maps $\overline{P}_{k_1}$ into P_{k_1} and $\overline{P}_{k_3}$ into P_{k_3}. Next, we show that $\Big\{x \in P\Big(S, k_2, \dfrac{k_2}{\eta}\Big) : S(x) > k_2\Big\} \neq \emptyset$ and $S(Tx) > k_2$ for all $x \in P\Big(S, k_2, \dfrac{k_2}{\eta}\Big)$. In fact, the constant function

$$\frac{k_2 + \dfrac{k_2}{\eta}}{2} \in \Big\{x \in P\Big(S, k_2, \frac{k_2}{\eta}\Big) : S(x) > k_2\Big\}.$$

Moreover for $x \in P\bigg(S, k_2, \dfrac{k_2}{\eta}\bigg)$, we have

$$\frac{k_2}{\eta} \geq \|x\| \geq x(t) \geq \min_{t\in[\tau,1]} \Big\{x(t)\Big\} = S(x) \geq k_2,$$

for all $t \in [\tau, 1]$. Thus, in view of $(H2)$ we see that

$$\begin{aligned} S\big(x(t)\big) &= \min_{t\in[\tau,1]} \left\{ \int_0^1 H(t,s) f\big(s, x(s)\big) ds \right\} \\ &\geq \min_{t\in[\tau,1]} \left\{ \int_\tau^1 H(t,s) f\big(s, x(s)\big) ds \right\} \\ &> \frac{k_2}{\mathcal{M}} \max_{t\in[0,1]} \left\{ \int_0^1 H(t,s) ds \right\} \\ &= k_2, \end{aligned}$$

as required. Finally, we show that $S(Tx) > k_2$ if $x \in P(S, k_2, k_3)$ and $\|Tx\| > \frac{k_2}{\eta}$. For this, we suppose that $x \in P(S, k_2, k_3)$ and $\|Tx\| > \frac{k_2}{\eta}$. Then

$$\begin{aligned} S\big(Tx(t)\big) &= \min_{t\in[\tau,1]} \left\{ \int_0^1 H(t,s) f\big(s, x(s)\big) ds \right\} \\ &\geq \eta \int_0^1 H(1,s) f\big(s, x(s)\big) ds \\ &\geq \max_{t\in[0,1]} \left\{ \int_0^1 H(t,s) f\big(s, x(s)\big) ds \right\} \\ &> \frac{k_2}{\mathcal{M}} \max_{t\in[0,1]} \left\{ \int_0^1 H(t,s) ds \right\} \\ &= k_2. \end{aligned}$$

Thus, all the conditions of Theorem 3.2 are satisfied. Therefore, the fractional order boundary value problem (1.1)-(1.2) has at least three positive solutions w_1, w_2, w_3 such that

$$\|w_1\| < k_1, k_2 < \min_{t\in[\tau,1]} \big\{w_2\big\}, \|w_3\| > k_1, \min_{t\in[\tau,1]} \big\{w_2\big\} < k_2.$$

□

4. An Example

In this section, as an application, the result is demonstrated with an example.

Consider the fractional order three-point boundary value problem

$$D_{0^+}^{2.9} x(t) + f\big(t, x\big) = 0, \; t \in (0,1), \tag{4.1}$$

$$\left.\begin{aligned} & x(0) = 0, \quad x'(0) = 0, \\ & \frac{15}{2} D_{0^+}^{1.7} x(1) = \frac{7}{2} D_{0^+}^{1.7} x\left(\frac{1}{2}\right), \end{aligned}\right\} \tag{4.2}$$

where

$$f(t,x) = \begin{cases} \dfrac{20}{3}\Big[x^2 - x\Big] + \dfrac{1}{153}(1-t^2)^{\frac{1}{2}} + \dfrac{1}{187}, & x \in [0,3], \\[2ex] 8\Big[2 \; log_3 x + x\Big] + \dfrac{1}{153}(1-t^2)^{\frac{1}{2}} + \dfrac{1}{187}, & x \in (3, \infty). \end{cases}$$

Clearly the function f is continuous and increasing on $[0, \infty)$. By direct calculations, one can get $\eta = 0.02767, \mathcal{M} = 0.095643$, $\mathcal{N} = 0.071238$. If we choose $k_1 = 1.5, k_2 = 2.05$ and $k_3 = 1500$. Then, all the conditions of Theorem 3.2 are satisfied. Therefore, by Theorem 3.2, the fractional order boundary value problem (4.1)-(4.2) has at least three positive solutions.

Acknowledgement

The author expresses his deep sense of gratitude to his guide Dr. K. R. Prasad, Professor of Applied Mathematics, Andhra University, Visakhapatnam.

References

[1] R. P. Agarwal, D. O'Regan, P. J. Y. Wong, *Positive solutions of differential, difference and integral equations*, Kluwer Academic Publishers, Dordrecht, (1999).

[2] B. Ahmad, J. J. Nieto, *Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions*, Comput. Math. Appl., **58** (2009), 1838–1843. 1

[3] Z. Bai, H. Lü, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl., **311** (2005), 495–505. 1

[4] M. Benchohra, J. Henderson, S. K. Ntouyas, A. Ouahab, *Existence results for fractional order functional differential equations with infinite delay*, J. Math. Anal. Appl., **338** (2008), 1340–1350. 1

[5] K. Diethelm, N. J. Ford, *Analysis of fractional differential equations*, J. Math. Anal. Appl., **265** (2002), 229–248.

[6] E. R. Kauffman, E. Mboumi, *Positive solutions of a boundary value problem for a nonlinear fractional differential equation*, Elec. J. Qual. Theory Diff. Equ., **3** (2008), 1–11. 1

[7] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204. Elsevier Science B. V., Amsterdam, (2006).

[8] R. W. Leggett, L. R. Williams, *Multiple positive fixed points of nonlinear operator on order Banach spaces*, Indiana Uni. Math. J., **28** (1979), 673–688. 2.3

[9] K. S. Miller, B. Ross, *An Introduction to fractional calculus and fractional differential equations*, John Wiley & Sons, Inc., New York, (1993).

[10] I. Podlubny, *Fractional diffrential equations*, Academic Press, Inc., San Diego, CA, (1999).

[11] K. R. Prasad, B. M. B. Krushna, *Multiple positive solutions for a coupled system of Riemann–Liouville fractional order two-point boundary value problems*, Nonlinear Stud., **20** (2013), 501–511. 1

[12] K. R. Prasad, B. M. B. Krushna, *Eigenvalues for iterative systems of Sturm–Liouville fractional order two-point boundary value problems*, Fract. Calc. Appl. Anal., **17** (2014), 638–653. 1

[13] K. R. Prasad, B. M. B. Krushna, N. Sreedhar, *Eigenvalues for iterative systems of (n, p)-type fractional order boundary value problems*, Int. J. Anal. Appl., **5** (2014), 136–146. 1

[14] K. R. Prasad, B. M. B. Krushna, *Lower and upper solutions for general two-point fractional order boundary value problems*, TWMS J. App. Eng. Math., **5** (2015), 80–87. 1