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ε-continuity in the Hyers–Ulam–Rassias stability

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Abstract

We improve the well-known Hyers–Ulam–Rassias stability of the linear mappings in Banach spaces of Th. M. Rassias, by relaxing the assumption of continuity. We also state some results concerned with the stability of ε-continuous functions.

Keywords: ε-continuous function, Hyers–Ulam–Rassias stability, approximate additive mapping.
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1. Introduction

In this note, we deal with the well-known problem that was raised in a famous talk presented by Stanislaw M. Ulam in 1940 in connection with the study of stability problem for homomorphisms. The problem is about finding conditions for a linear mapping to be close enough to a given approximately linear mapping. Donald H. Hyers [3] was able to provide a partial answer to Ulam's problem in 1941 which was, undoubtedly, the first significant effort in the issue. Finally, Themistocles M. Rassias extended the result of Hyers; [13]. He proved for function f, defined between Banach spaces, (p, ε)-additive in the sense introduced below, and satisfying in a type of local continuity, we have got the unique solution for the problem. The issue has received much attention through the last few decades; [1, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16]. It has to be stated that this result, independently, was considered by T. Aoki [2] for about 70 years ago which has not attracted real attention since then. In this note, we show that the condition of continuity in this result, could be replaced by a type of ε-continuity, which is obviously weaker than that type of continuity considered by Rassias, and the conclusion of the theorem is still going on. We emphasize here that the very bad behavioral Dirichlet function satisfies the conditions of our theorem. So we really improve the result stated by Rassias. Then we state some results related to the stability problem of ε-continuous real functions.

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Let $\mathscr{X}_1$ and $\mathscr{X}_2$ be Banach spaces and $f : \mathscr{X}_1 \to \mathscr{X}_2$ a mapping and let $\varepsilon > 0$ and $0 \leq p < \infty$. Mapping f is said to be (p, ε)-additive if

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p),$$

for all $x, y \in \mathscr{X}_1$; see [13, 17]. Mapping f is called ε-continuous at $x \in \mathscr{X}_1$, if for all $\varepsilon_1 > \varepsilon$, there exists $\delta > 0$ so that $\|f(x) - f(y)\| < \varepsilon_1$, whenever $\|x - y\| < \delta$. For Banach space $\mathscr{X}$, the notation $\mathscr{X}^*$ is used to denote the dual space of $\mathscr{X}$, consisting of all bounded linear functionals on this space.

2. Main results

In this section we express our results related to the well-known Hyers–Ulam–Rassias stability for linear mappings in Banach spaces. Our method may be applied to other types of stabilities involving continuous functions. We, firstly, in the following theorem, state a relationship between ε-continuity and additivity. Our main result, in this paper, requires its statement. Its proof is immediate from a well-known result, namely, if f is locally bounded at 0 and additive whence it must be continuous. But for the sake of completeness we state its proof.

Theorem 2.1. *Let $f : \mathscr{X} \to \mathscr{Y}$ be an additive mapping between normed spaces and $\varepsilon_0 > 0$. If f is ε_0-continuous at 0, then it is continuous on $\mathscr{X}$.*

Proof. First we note that since f is additive, $f(nx) = nf(x)$ for all $n \in \mathbb{N}$. Thus $f(\frac{n}{m}x) = \frac{n}{m}f(x)$ for all $m, n \in \mathbb{N}$. Hence $f(rx) = rf(x)$ for all rational number r. Let $\varepsilon > \varepsilon_0$. There exist $\delta > 0$ so that, if $\|x\| < \delta$ then $\|f(x)\| < \varepsilon$. We can choose δ in $\mathbb{Q}$. Now, let $x \in \mathscr{X}$ and $y = \frac{\delta x}{2h}$ where $h \in \mathbb{Q}$ is chosen in the way that satisfies

$$\frac{\|x\|}{2} < h < \|x\|.$$

Therefore $\|y\| < \delta$ which implies that $\|f(y)\| = \|f(\frac{\delta x}{2h})\| < \varepsilon$. But $\frac{\delta}{2h} \in \mathbb{Q}$ so $f(\frac{\delta x}{2h}) = \frac{\delta}{2h}f(x)$. It follows that

$$\|f(x)\| < \frac{2\varepsilon h}{\delta} < \frac{2\varepsilon\|x\|}{\delta}.$$

Thus by additivity of f we have that

$$\|f(x) - f(y)\| = \|f(x-y)\| < \frac{2\varepsilon\|x-y\|}{\delta},$$

which establishes the continuity of f. □

In this point we express our results related to Hyers–Ulam–Rassias stability for linear mappings.

Theorem 2.2. *Let ε_1 and ε_2 be positive real numbers and $0 \leq p < 1$, and let $f : \mathscr{X}_1 \to \mathscr{X}_2$ be a (p, ε_1)-additive mapping between Banach spaces. If $f(tx)$ is ε_2-continuous in parameter t at 0 for each x, then there exists a unique linear mapping $T : \mathscr{X}_1 \to \mathscr{X}_2$ such that*

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon_1}{2 - 2^p}\|x\|^p,$$

for all $x \in \mathscr{X}_1$.

Proof. Let $\varepsilon > \max\{\varepsilon_1, \varepsilon_2\}$ and $x \in \mathscr{X}_1$ be fixed. There exists $\varepsilon > \delta > 0$ so that if $|t| < \delta$, then $\|f(tx) - f(0)\| < \varepsilon$. Let x_0 be in $\mathscr{X}_1$ with $\|x_0\| = 1$. From

$$\frac{\|f(x_0) - f(x_0) - f(0)\|}{\|x_0\|^p} < \varepsilon,$$

we have that $\|f(0)\| < \varepsilon$. Note that in the cases when $p \neq 0$ we could conclude that $f(0) = 0$. So if $|t| < \delta$, then $\|f(tx)\| \leq 2\varepsilon$.

On the other hand from [13] we know that there exists an additive mapping T so that

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon_1}{2 - 2^p}\|x\|^p,$$

for all $x \in \mathscr{X}_1$. We want to show that T is linear. For, we merely need to show that $T(tx)$ is continuous in parameter t; see [13]. But if $|t| < \delta$, then

$$\begin{aligned} \|T(tx)\| &= \|T(tx) - f(tx) + f(tx)\| \leq \|T(tx) - f(tx)\| + \|f(tx)\| \\ &< \theta|t|^p\|x\|^p + 2\varepsilon < \theta\varepsilon^p\|x\|^p + 2\varepsilon, \end{aligned}$$

where $\theta = \frac{2\varepsilon_1}{2-2^p}$. Hence for $\rho \in \mathscr{X}_2^*$ we have that

$$\|\rho(T(tx))\| < \|\rho\|(\theta\varepsilon^p\|x\|^p + 2\varepsilon),$$

whenever $|t| < \delta$. This ensures that the additive mapping $\phi(t) = \rho(T(tx))$ is ε_0-continuous at 0 where $\varepsilon_0 = \|\rho\|\,(\theta\varepsilon^p\|x\|^p + 2\varepsilon)$. Thus it is continuous in parameter t by the previous theorem. □

We could restate Hyers' theorem as well,

Corollary 2.3. *Let ε_1 and ε_2 be positive real numbers, and let $f : \mathscr{X}_1 \to \mathscr{X}_2$ be a mapping between Banach spaces so that $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$. If $f(tx)$ is ε_2-continuous in parameter t at 0 for each x, then there exists a unique linear mapping $T : \mathscr{X}_1 \to \mathscr{X}_2$ such that*

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon_1}{2 - 2^p},$$

for all $x \in \mathscr{X}_1$.

We are now led to the following strengthening of main result of [14], relaxing the cumbersome restriction of continuity. Its proof is comparatively the same as the previous theorem so it is somehow unnecessary to state.

Theorem 2.4. *Let ε_1 and ε_2 be positive real numbers, and let $f : \mathscr{X}_1 \to \mathscr{X}_2$ be a mapping between Banach spaces so that $\|f(x + y) - f(x) - f(y)\| \leq 2\varepsilon_1\|x\|^a\|y\|^a$ for some $0 \leq a < \frac{1}{2}$ and for any $x, y \in \mathscr{X}_1$. If $f(tx)$ is ε_2-continuous in parameter t at 0 for each x, then there exists a unique linear mapping $T : \mathscr{X}_1 \to \mathscr{X}_2$ such that*

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon_1\|x\|^{2a}}{1 - 2^{2a-1}},$$

for all $x \in \mathscr{X}_1$.

Remark 2.5. Re-examining the proof of the recent theorems, we see that if whichever of two functions f and g, is ε_1-continuous for some $\varepsilon_1 > 0$, and if $\|f(x) - g(x)\| < \varepsilon_0$ for given $\varepsilon_0 > 0$, then there exists an $\varepsilon_2 > 0$ such that the other function is ε_2-continuous. This theme suggests speaking about the stability problem of the continuous functions. In the following we study this problem.

Theorem 2.6. *Let $\varepsilon > 0$ and $f : [0, 1] \to \mathbb{R}$ be a ε-continuous function on $[0, 1]$. Then there exists a continuous function g on this interval so that*

$$\|f - g\|_\infty \leq \varepsilon, \tag{2.1}$$

where $\|\cdot\|_\infty$ stands for the supremum norm.

Proof. Due to the compactness of [0,1], it is readily verified that f is uniformly ε-continuous, which means there exists $\delta > 0$ so that $|f(x) - f(y)| \leq \varepsilon$ whenever $|x - y| < \delta$. Choose $k \in \mathbb{N}$ with $\frac{1}{k} < \delta$ and consider the set $\{0, \frac{1}{k}, \frac{2}{k}, \cdots, \frac{k-1}{k}, 1\}$ as a partition of interval [0,1]. On the sth subinterval define function g_s to be

$$g_s(x) := k \left(f(\frac{s}{k}) - f(\frac{s-1}{k}) \right) (x - \frac{s}{k}) + f(\frac{s}{k}),$$

which describes in fact a line segment between points $(\frac{s}{k}, f(\frac{s}{k}))$ and $(\frac{s-1}{k}, f(\frac{s-1}{k}))$ in $\mathbb{R}^2$. For $x \in [\frac{s-1}{k}, \frac{s}{k}]$, there exists $\lambda \in [0,1]$ so that $g_s(x) = \lambda f(\frac{s-1}{k}) + (1-\lambda) f(\frac{s}{k})$, wherefrom

$$|g_s(x) - f(x)| = \left| \lambda f(\frac{s-1}{k}) + (1-\lambda) f(\frac{s}{k}) - \lambda f(x) - (1-\lambda) f(x) \right| \leq \varepsilon.$$

Define g to be

$$g(x) = \sum_{i=1}^{n} \chi_{[\frac{i-1}{k}, \frac{i}{k}]} g_i(x),$$

where χ_A is the characteristic function defined to be $\chi_A(x) = \begin{cases} 0 & x \notin A \\ 1 & x \in A \end{cases}$. Now, it is easy to see that g is continuous and $\|f - g\|_\infty < \varepsilon$. $\square$

We generalize this theorem to the functions defined across the allover real axis as follows,

Corollary 2.7. *If $f : \mathbb{R} \to \mathbb{R}$ is an ε-continuous function, then there exists a continuous function g with $\|f - g\|_\infty < \varepsilon$.*

Proof. Consider $\mathbb{R} = \cup_{i \in \mathbb{Z}} [i-1, i]$. Applying the previous theorem on $I_i := [i-1, i]$, we reach sequence $\{g_i\}_{i \in \mathbb{Z}}$. Define g to be

$$g = \sum_{i \in \mathbb{Z}} \chi_{I_i} g_i,$$

and we are done. $\square$

Clearly, function g in Theorem 2.6 is not determined uniquely. Number ε in (2.1) also does not give us the best possible choice for the function g. For example, let

$$D(x) = \begin{cases} 0, & x \in \mathbb{Q}, \\ 1, & x \notin \mathbb{Q}, \end{cases}$$

be Dirichlet function. This function is 1-continuous. For every continuous function g with $g(x) \in [0,1]$, we have that $\|f - g\|_\infty < 1$. However, it seems that the function g_0 defined for any x to be $g_0(x) = \frac{1}{2}$, has the slightest distance with D, which is $\frac{1}{2}$, and g_0 is unique satisfying this distance circumstance. On the other hand, D satisfies the conditions of Theorem 2.2, and T in this theorem is given by $T \equiv 0$ which is determined uniquely due to the theorem. But $\|D - T\|_\infty = \frac{1}{2}$ and we have no idea whichever is better, g_0 or T.

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