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Fixed point theorems for Geraghty contractions in partially ordered partial b - metric spaces

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Abstract

In this paper we consider the concept of generalized Geraghty contractive self mapping in a complete partially ordered partial b-metric space. We study the existence of fixed points for such a self mapping in complete partially ordered partial b-metric spaces controlled by generalized Geraghty contractive type condition and obtain some fixed point results of [V. La Rosa, P. Vetro, J. Nonlinear Sci. Appl., **7** (2014), 1–10] in complete partially ordered partial b-metric spaces as corollaries. Supporting example is also provided.

Keywords: Fixed point theorems, contractive mappings, partial metric, partial b-metric, ordered partial metric space, partially ordered partial b-metric space, Geraghty contraction.
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1. Introduction

Fixed point theorems usually start from Banach [5] contraction principle. But all the generalizations may not be from this principle. In 1973, Geraghty [7] introduced an extension of the contraction in which the contraction constant was replaced by a function having some specified properties. In 1989, Bakktin [4] introduced the concept of a b-metric space as a generalization of a metric spaces. In 1993, Czerwik [6] extended many results related to the b-metric spaces. In 1994, Matthews [12] introduced the concept of partial metric space in which the self distance of any point of space may not be zero. In 1996, O'Neill [17]

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generalized the concept of partial metric space by admitting negative distances. Moreover, the existence of several connections between partial metrics and topological aspects of domain theory have been pointed by O'Neill [17]. In 2013, Shukla [21] generalized both the concepts of b-metric and partial metric space by introducing the partial b-metric spaces. Many authors recently studied the existence of fixed points of self maps in different types of metric spaces [1, 3, 8, 16, 19, 20, 21]. Xian Zhang [23] proved a common fixed point theorem for two self maps on a metric space satisfying generalized contractive type conditions. Some authors studied some fixed point theorems in b-metric spaces [13, 22, 24]. After that some authors started to prove $\alpha - \psi$ versions of certain fixed point theorems in different type metric spaces [2, 9, 10]. Mustafa[15] gave a generalization of Banach contraction principle in complete ordered partial b-metric space by introducing a generalized $\alpha - \psi$ weakly contractive mapping. Aiman Mukheimer [14] generalized the concept of Mustafa [15] by introducing the α-ψ-φ contractive mapping in a complete ordered partial b-metric space.

In this paper we prove fixed point theorems for generalized Geraghty contractive self mappings in complete partially ordered partial b-metric spaces satisfying a contractive type condition by considering partial b-metric p as in Definition 2.1 (Shukla [21]) which is more general than that of any partial b-metric. We also obtained some fixed point results of V. La Rosa et al. [11] in complete partially ordered partial b-metric space as corollaries .

A supporting example is given and an open problem is also given at the end of the paper. Shukla [21] introduced the notation of a partial b-metric space as follows.

2. Preliminaries

We first offer several basic facts used throughout this paper.

Definition 2.1 ([21]). Let X be a non empty set and let $s \geq 1$ be a given real number. A function $p : X \times X \to [0, \infty)$ is called a partial b-metric if for all $x, y, z \in X$, the following conditions are satisfied.

(i) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$;

(ii) $p(x, x) \leq p(x, y)$;

(iii) $p(x, y) = p(y, x)$;

(iv) $p(x, y) \leq s\{p(x, z) + p(z, y)\}$ - $p(z, z)$.

The pair (X, p) is called a partial b-metric space. The number $s \geq 1$ is called a coefficient of (X, p).

Definition 2.2 ([9]). Let $(X, \leq)$ be a partially ordered set and $f : X \to X$ be a mapping. We say that f is non decreasing with respect to $'' \leq''$ if $x, y \in X$, $x \leq y \Rightarrow fx \leq fy$.

Definition 2.3 ([9]). Let $(X, \leq)$ be a partially ordered set. A sequence $\{x_n\} \in X$ is said to be non decreasing with respect to $'' \leq''$ if $x_n \leq x_{n+1} \forall\ n \in N$.

Definition 2.4 ([15]). A triple $(X, \leq, p)$ is called an ordered partial b - metric space if $(X, \leq)$ is a partially ordered set and p is a partial b-metric on X.

Definition 2.5 ([7]). A self map $f : X \to X$ is said to be a Geraghty contraction if there exists $\beta \in S$ such that

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y),$$

where $S = \{\beta : [0, \infty) \to [0, 1)/\beta(t_n) \to 1 \Rightarrow t_n \to 0\}$.

Definition 2.6 ([18])**.** Suppose $(X, \leq, p)$ is a partially ordered partial b-metric space and $f : X \to X$ is a self map. Let $\alpha : X \times X \to [0, +\infty)$. f is said to be $\alpha-$ admissible if for all $x, y \in X$, $\alpha(x, y) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1$.

Definition 2.7 ([11])**.** Let $(X, \leq)$ be a partially ordered set and suppose that there exists a partial metric p such that (X, p) is a partial metric space. Let f be a self mapping on X. If there exists $\beta \in S$ such that $p(f(x), f(y)) \leq \beta(M(x, y))M(x, y)$ for all $x, y \in X$ where

$$M(x, y) = max\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2}[p(x, fy) + p(fx, y)]\},$$

then we say that f is a generalized Geraghty contraction map.

Definition 2.8 ([11])**.** Let $(X, \leq)$ be a partially ordered set and suppose that there exists a partial metric p such that (X, p) is a partial metric space. Let $\alpha : X \times X \to [0, +\infty)$. X is called $\alpha-$regular if for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geq 1 \ \forall \ n \in N \cup \{0\}$ and $x_n \to x$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1 \ \forall \ k \in N$.

V. La Rosa et a.l [11] proved the following theorems.

Theorem 2.9 ([11], Theorem 3.5)**.** *Let $(X, \leq, p)$ be a complete partial metric space and let $\alpha : X \times X \to [0, \infty)$ be a function. Let $f : X \to X$ be a self mapping. Suppose that there exists $\beta \in S$ such that $\alpha(x, fx)\alpha(y, fy)p(fx, fy) \leq \beta(M(x, y))M(x, y)$ for all $x, y \in X$, where*

$$M(x, y) = max\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2}[p(x, fy) + p(fx, y)]\}.$$

Assume that

(i) f is α admissible,

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$,

(iii) for every sequence $\{x_n\}$ in X such that $\alpha(x_n, fx_n) \geq 1 \ \ \forall \ n \in N \cup \{0\}$ and x_n converges to x, then $\alpha(x, fx) \geq 1$,

(iv) $\alpha(x, fx) \geq 1 \ \forall \ x \in Fix(f)$.

Then f has a unique fixed point z in X.

Theorem 2.10 ([11], Theorem 3.6)**.** *Let $(X, \leq, p)$ be a complete partial metric space and let $\alpha : X \times X \to [0, \infty)$ be a function. Let $f : X \to X$ be a self mapping. Suppose that there exists $\beta \in S$ such that $\alpha(x, y)p(fx, fy) \leq \beta(M(x, y))M(x, y)$ for all $x, y \in X$, where*

$$M(x, y) = max\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2}[p(x, fy) + p(fx, y)]\}.$$

Assume that

(i) f is α admissible,

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$,

(iii) X is $\alpha-$regular and for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geq 1 \ \ \forall \ n \in N \cup \{0\}$, we have $\alpha(x_m, x_n) \geq 1$ for all $m, n \in N$ with $m < n$,

(iv) $\alpha(x, y) \geq 1 \ \forall \ x, y \in Fix(f)$.

Then f has a unique fixed point $z \in X$.

Theorem 2.11 ([11], Theorem 4.1)**.** *Let $(X, \leq, p)$ be a complete ordered partial metric space and let $\alpha : X \times X \to [0, \infty)$ be a function. Let $f : X \to X$ be a non-decreasing mapping. Suppose that there exists $\beta \in S$ such that*

$$p(fx, fy) \leq \beta(M(x, y))M(x, y) \text{ for all } x, y \in X \text{ with } x \leq y,$$

where

$$M(x, y) = max\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2}[p(x, fy) + p(y, fx)]\}.$$

Assume also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \leq fx_0$,

(ii) X is such that, if a non-decreasing sequence $\{x_n\}$ converges to x, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\} \leq x \ \forall \ k \in N$,

(iii) x, y are comparable whenever $x, y \in Fix(f)$.

Then f has a unique fixed point $z \in X$.

3. Main results

In this section we extend the study of Theorems 2.9, 2.10 and 2.11 for partially ordered partial b-metric spaces by using partial b-metric p of Definition 2.1 . We begin this section with the following definition.

Definition 3.1. Suppose $(X, \leq)$ be a partially ordered set and p be a partial b-metric in the sense of Definition 2.1 with $s \geq 1$ as the coefficient of (X, p). Then we say that the triplet $(X, \leq, p)$ is a partially ordered partial b-metric space. A partially ordered partial b-metric space $(X, \leq, p)$ is said to be complete if every Cauchy sequence in X is convergent in the sense of the Definition 2.1. We observe that every ordered partial b-metric space is a partially ordered partial b-metric space, in the light of the observation made above.

Definition 3.2. Let $(X, \leq)$ be a partially ordered set and suppose that there exists a partial b-metric p such that (X, p) is a partial b-metric space with coefficient $s \geq 1$. Let f be a self mapping on X. If there exists $\beta \in S$ such that $sp(f(x), f(y)) \leq \beta(M(x, y))M(x, y)$ for all $x, y \in X$ where $M(x, y) = max\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2s}[p(x, fy) + p(fx, y)]\}$, then we say that f is a generalized Geraghty contraction map.

Now we state the following useful lemmas, whose proofs can be found in Sastry et al. [20].

Lemma 3.3. *Let $(X, \leq, p)$ be a p complete partially ordered partial b-metric space with coefficient $s \geq 1$. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$. Suppose $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$. Then $\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, y) = p(x, y)$ and hence $x = y$.*

Lemma 3.4.

(i) $p(x, y) = 0 \Rightarrow x = y$,

(ii) $\lim_{n\to\infty} p(x_n, x) = 0 \Rightarrow p(x, x) = 0$ and hence $x_n \to x$ as $n \to \infty$.

Lemma 3.5. *Let $(X, \leq, p)$ be a partially ordered partial b-metric space with coefficient $s \geq 1$. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$. Then*

(i) $\{x_n\}$ is a Cauchy sequence $\Rightarrow \lim_{m,n\to\infty} p(x_m, x_n) = 0$,

(ii) $\{x_n\}$ *is not a Cauchy sequence* $\Rightarrow \exists\ \epsilon > 0$ *and sequences* $\{m_k\}$, $\{n_k\} \ni m_k > n_k > k \in N$; $p(x_{n_k},x_{m_k}) > \epsilon$ *and* $p(x_{n_k},x_{m_k-1}) \leq \epsilon$.

Proof. (i) Suppose $\{x_n\}$ is a Cauchy sequence, then $\lim\limits_{m,n\to\infty} p(x_m, x_n)$ exists and is finite. Therefore $0 = \lim\limits_{n\to\infty} p(x_n, x_{n+1}) = \lim\limits_{m,n\to\infty} p(x_m, x_n)$. Therefore $\lim\limits_{m,n\to\infty} p(x_m, x_n) = 0$.
(ii) $\{x_n\}$ is not a Cauchy sequence $\Rightarrow \lim\limits_{m,n\to\infty} p(x_m, x_n) \neq 0$ if it exists
$\Rightarrow \exists\ \epsilon > 0$ and for every N and $m, n > N \ni p(x_m, x_n) > \epsilon$

$$\because \lim_{n\to\infty} p(x_n, x_{n+1}) = 0 \Rightarrow \exists\ M \ni p(x_n, x_{n+1}) < \epsilon\ \forall\ n > M.$$

Let $N_1 > M$ and n_1 be the smallest such that $m > n_1$ and $p(x_{n_1},x_m) > \epsilon$ for at least one m. Let m_1 be the smallest such that $m_1 > n_1 > N_1 > 1$ and $p(x_{n_1},x_{m_1}) > \epsilon$ so that $p(x_{n_1},x_{m_1-1}) \leq \epsilon$. Let $N_2 > N_1$ and choose $m_2 > n_2 > N_2 > 2 \ni p(x_{n_2},x_{m_2}) > \epsilon$ and $p(x_{n_2},x_{m_2-1}) \leq \epsilon$.
Continuing this process we can get sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that $m_k > n_k > k$ and $p(x_m,x_n) > \epsilon$; $p(x_{n_k},x_{m_k-1}) \leq \epsilon$. □

Lemma 3.6. *Let* $(X, \leq, p)$ *be a partially ordered partial b-metric space with coefficient* $s \geq 1$. *Let* $\{x_n\}$ *be a sequence in* X *such that* $sp(x_n, y) \leq p(x, y)$ *and* $\{x_n\} \to x$ *as* $n \to \infty$, *then* $\{sp(x_n, y)\} \to p(x, y)$ *as* $n \to \infty$.

Proof. Since $sp(x_n, y) \leq p(x, y)$, then $\limsup\limits_{n\to\infty} sp(x_n, y) \leq p(x, y)$. On the other hand

$$\begin{aligned}
p(x, y) &\leq sp(x, x_n) + sp(x_n, y) - p(x_n, x_n)\\
&\leq sp(x, x_n) + sp(x_n, y)\\
&\Rightarrow p(x, y) \leq \liminf_{n\to\infty} sp(x_n, y)\\
&\therefore \limsup_{n\to\infty} sp(x_n, y) \leq p(x, y) \leq \liminf_{n\to\infty} sp(x_n, y)\\
&\therefore \lim_{n\to\infty} sp(x_n, y) = p(x, y)
\end{aligned}$$

□

Now we state our first main result :

Theorem 3.7. *Let* $(X, \leq, p)$ *be a complete partially ordered partial b-metric space with* $s \geq 1$ *and let* $\alpha : X \times X \to [0, \infty)$ *be a function. Let* $f : X \to X$ *be a self map. Suppose that there exists* $\beta \in S$ *such that* $\alpha(x, fx)\alpha(y, fy)sp(fx, fy) \leq \beta(M(x, y))M(x, y)$ *for all* $x, y \in X$, *where*

$$M(x, y) = max\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2s}[p(x, fy) + p(fx, y)]\}. \tag{3.1}$$

Assume that

(i) f *is* α *admissible,*

(ii) there exists $x_0 \in X$ *such that* $\alpha(x_0,fx_0) \geq 1$,

(iii) for every sequence $\{x_n\}$ *in* X *such that* $\alpha(x_n,fx_n) \geq 1\ \ \forall\ n \in N \cup \{0\}$ *and* x_n *converges to* x, *then* $\alpha(x, fx) \geq 1$,

(iv) $\alpha(x, fx) \geq 1\ \forall\ x \in Fix(f)$.

Then f *has a unique fixed point* z *in* X.

Proof. By (ii), let $x_0 \in X$ such that $\alpha(x_0,fx_0) \geq 1$. We choose $x_n \in X$ such that $x_n = fx_{n-1}\ \forall\ n \in N$. Since f is α - admissible, we have

$$\alpha(fx_0, fx_1) = \alpha(x_1, x_2) \geq 1,\ \alpha(fx_1, fx_2) = \alpha(x_2, x_3) \geq 1.$$

By induction, we get

$$\alpha(x_n,\ x_{n+1}) \geq 1 \text{ for all } n \in N \cup \{0\}. \tag{3.2}$$

If $x_n = x_{n+1}$ for some $n \in N \cup \{0\}$, then $x_n = x_{n+1} = fx_n$ and so x_n is a fixed point of f. Hence we may assume that $x_n \neq x_{n+1}\ \forall n \in N$. Then we have $p(x_n, x_{n+1}) > 0$, therefore by (3.1)

$$\begin{aligned} sp(x_{n+1}, x_{n+2}) &= sp(fx_n, fx_{n+1}) \\ &\leq \alpha(x_n, fx_n)\alpha(x_{n+1},\ f_{n+1})sp(fx_n, fx_{n+1}) \\ &\leq \beta(M(x_n, x_{n+1}))M(x_n, x_{n+1}) \\ &< M(x_n,\ x_{n+1}), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} M(x_n, x_{n+1}) &= max\{p(x_n, x_{n+1}), p(x_{n+1}, fx_{n+1}), p(x_n, fx_n), \frac{1}{2s}[p(x_{n+1}, fx_n) + p(x_n, fx_{n+1})]\} \\ &= max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), p(x_n, x_{n+1}), \frac{1}{2s}[p(x_{n+1}, x_{n+1}) + p(x_n, x_{n+2})]\} \\ &\leq max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), p(x_n, x_{n+1}), \\ &\quad \frac{1}{2s}[p(x_{n+1}, x_{n+1}) + s(p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2})) - p(x_{n+1}, x_{n+1})] \\ &= max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), p(x_n, x_{n+1}), \frac{1}{2s}[s(p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}))] \\ &= max[p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})]. \end{aligned} \tag{3.4}$$

Suppose

$$M(x_n, x_{n+1}) = p(x_{n+1}, x_{n+2}). \tag{3.5}$$

Then by (3.3) $sp(x_{n+1},\ x_{n+2}) < p(x_{n+1}, x_{n+2})$, which is a contradiction.

$$\therefore M(x_n, x_{n+1}) = p(x_n, x_{n+1}) \tag{3.6}$$

$$\begin{aligned} \therefore p(x_{n+1}, x_{n+2}) &\leq sp(x_{n+1}, x_{n+2}) \\ &\leq \alpha(x_n, fx_n)\alpha(x_{n+1}, f_{n+1})sp(fx_n, fx_{n+1}) \\ &\leq \beta(p(x_n, x_{n+1})p(x_n, x_{n+1}) \\ &< p(x_n, x_{n+1}), \end{aligned}$$

therefore, sequence $\{p(x_n, x_{n+1}))\}$ is strictly decreasing and converges to r (say).

Suppose $r \neq 0$, then

$$\frac{p(x_{n+1}, x_{n+2})}{p(x_n, x_{n+1})} \leq \beta(p(x_n, x_{n+1})) < 1. \tag{3.7}$$

Taking limits as $n \to \infty$,

$$\begin{aligned} &\therefore \lim_{n\to\infty} \beta(p(x_n, x_{n+1})) = 1 \Rightarrow \lim_{n\to\infty} p(x_n, x_{n+1}) = 0, \\ &\therefore r = \lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \end{aligned} \tag{3.8}$$

Now we claim sequence $\{x_n\}$ is a Cauchy sequence. Assume that $\{x_n\}$ is not a Cauchy sequence. Then by Lemma 3.5 $\exists\ \epsilon > 0$ and sequences $\{x_{n_k}\}$, $\{x_{m_k}\}$; $m_k > n_k > k$ such that $p(x_{m_k}, x_{n_k}) \geq \epsilon$ and $p(x_{m_k-1}, x_{n_k})$ $< \epsilon$.

Therefore

$$\begin{aligned}
\epsilon &\le sp(x_{m_k}, x_{n_k}) \\
&= sp(fx_{m_k-1}, fx_{n_k-1})\} \\
&\le \alpha(x_{m_k-1}, fx_{m_k-1})\alpha(x_{n_k-1}, fx_{n_k-1})\beta(M(x_{m_k-1}, x_{n_k-1}))M(x_{m_k-1}, x_{n_k-1}) \\
&\le \beta(M(x_{m_k-1}, x_{n_k-1})M(x_{m_k-1}, x_{n_k-1}) < M(x_{m_k-1}, x_{n_k-1}),
\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}
&M(x_{m_k-1}, x_{n_k-1}) \\
&= max[p(x_{m_k-1}, x_{n_k-1}), p(x_{n_k-1}, fx_{n_k-1}), p(x_{m_k-1}, fx_{m_k-1}), \\
&\quad \frac{1}{2s}[\{p(x_{m_k-1}, fx_{n_k-1}) + p(fx_{m_k-1}, x_{n_k-1})\}] \\
&= max[p(x_{m_k-1}, x_{n_k-1}), p(x_{n_k-1}, x_{n_k}), p(x_{m_k-1}, x_{m_k}), \frac{1}{2s}[\{p(x_{m_k-1}, x_{n_k}) + p(x_{m_k}, x_{n_k-1})\}] \\
&\le max[p(x_{m_k-1}, x_{n_k-1}), p(x_{n_k-1}, x_{n_k}), p(x_{m_k-1}, x_{m_k}), \\
&\quad \frac{1}{2s}[\{sp(x_{m_k-1}, x_{n_k-1}) + sp(x_{n_k-1}, x_{n_k}) - p(x_{n_k-1}, x_{n_k-1}) + sp(x_{m_k-1}, x_{n_k-1}) \\
&\qquad + sp(x_{m_k-1}, x_{m_k}) - p(gx_{m_k-1}, x_{m_k-1})\}] \\
&\le max[p(x_{m_k-1}, x_{n_k-1}), p(x_{n_k-1}, x_{n_k}), p(x_{m_k-1}, x_{m_k}), \\
&\quad \frac{1}{2s}[\{2sp(x_{m_k-1}, x_{n_k-1}) + sp(x_{n_k-1}, x_{n_k}) + sp(x_{m_k}, x_{m_k-1})\}] \\
&= p(x_{m_k-1}, x_{n_k-1}) + \frac{1}{2}p(x_{n_k-1}, x_{n_k}) + \frac{1}{2}p(x_{m_k}, x_{m_k-1}) \\
&\le sp(x_{m_k-1}, x_{n_k}) + sp(x_{n_k}, x_{n_k-1}) - p(x_{n_k}, x_{n_k}) + \frac{1}{2}p(x_{n_k-1}, x_{n_k}) + \frac{1}{2}p(x_{m_k}, x_{m_k-1}) \\
&\le sp(x_{m_k-1}, x_{n_k}) + sp(x_{n_k}, x_{n_k-1}) + \frac{1}{2}p(x_{n_k-1}, x_{n_k}) + \frac{1}{2}p(x_{m_k}, x_{m_k-1}) \\
&\le s\epsilon + s\eta + \frac{1}{2}\eta + \frac{1}{2}\eta,
\end{aligned}$$

where $p(x_{n_k-1},x_{n_k}) < \eta$ and $p(x_{m_k},x_{m_k-1}) < \eta$; $\eta \to 0$ as $k \to \infty$. Therefore

$$s\epsilon \le \beta(M(x_{m_k-1}, x_{n_k-1})(s\epsilon + s\eta + \eta). \tag{3.10}$$

Allowing $k \to \infty$, $s\epsilon \le \lim\limits_{k\to\infty} \beta(M(x_{m_k-1},x_{n_k-1}) \lim\limits_{k\to\infty} (s\epsilon + s\eta + \eta)$

$$\begin{gathered}
s\epsilon \le \lim_{k\to\infty} \beta(M(x_{m_k-1}, x_{n_k-1})(s\epsilon), \\
\therefore \lim_{k\to\infty} \beta(M(x_{m_k-1}, x_{n_k-1})) = 1, \\
\therefore \lim_{k\to\infty} M(x_{m_k-1}, x_{n_k-1}) = 0.
\end{gathered}$$

Then by (3.9) $s\epsilon \le 0$, which is a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence. Hence $\lim\limits_{n,m\to\infty} p(x_n, x_m)$ exists and is equal to 0 (by (3.8) and Lemma 3.5). Since (X, p) is complete,

$$\therefore \{x_n\} \to y \text{ for some } y \in X,$$

then

$$0 = \lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, y) = p(y, y) \text{ and } \alpha(y, fy) \ge 1 \text{ (by (iii))}. \tag{3.11}$$

Now,

$$\begin{aligned} sp(fx_n, fy) &\leq \alpha(x_n, fx_n)\alpha(y, fy)\beta(M(x_n, y))M(x_n, y) \\ &\leq \beta(M(x_n, y))M(x_n, y) \\ &< M(x_n, y), \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} M(x_n, y) &= max\{p(x_n, y), p(y, fy), p(x_n, fx_n), \frac{1}{2s}[p(x_n, fy) + p(fx_n, y)]\} \\ &= max\{p(x_n, y), p(y, fy), p(x_n, x_{n+1}), \frac{1}{2s}[p(x_n, fy) + p(x_{n+1}, y)]\} \\ &\leq max\{p(x_n, y), p(y, fy), p(x_n, x_{n+1}), \frac{1}{2s}[sp(x_n, y) + sp(y, fy) - p(y, y) + p(x_{n+1}, y)]\} \\ &= p(y, fy) \text{ for large } n. \end{aligned} \tag{3.13}$$

Therefore

$$sp(fx_n, fy) = sp(x_{n+1}, fy) < M(x_n, y) = p(y, fy).$$

But

$$\lim_{n\to\infty} x_{n+1} = y. \tag{3.14}$$

Therefore By Lemma 3.5,

$$\lim_{n\to\infty} sp(fx_n, fy) = p(y, fy). \tag{3.15}$$

Now by (3.12),

$$sp(fx_n, fy) \leq \beta(M(x_n, y))M(x_n, y) < M(x_n, y).$$

Allowing $n \to \infty$,

$$\begin{aligned} \lim_{n\to\infty} sp(fx_n, fy) &\leq \lim_{n\to\infty} \beta(M(x_n, y))M(x_n, y) \leq \lim_{n\to\infty} M(x_n, y) \\ \Rightarrow p(y, fy) &\leq \lim_{n\to\infty} \beta(M(x_n, y))p(y, fy) \leq p(y, fy). \end{aligned}$$

Therefore

$$\begin{aligned} &\lim_{n\to\infty} \beta(M(x_n, y)) = 1, \\ &\Rightarrow \lim_{n\to\infty} M(x_n, y) = 0, \\ &\Rightarrow p(y, fy) = 0 \Rightarrow y = fy. \end{aligned}$$

Therefore y is a fixed point of f.

Suppose y, z are distinct fixed points of f. Hence, $y, z \in Fix\{f\}$ and $p(y, z) > 0$. Therefore By (iv), $\alpha(y, fy) \geq 1$ and $\alpha(z, fz) \geq 1$. Then

$$\begin{aligned} p(y, z) &\leq sp(fy, fz) \\ &\leq \alpha(y, fy)\alpha(z, fz)sp(fy, fz) \\ &\leq \beta(M(y, z))M(y, z) \\ &< M(y, z), \end{aligned}$$

where

$$M(y, z) = max\{p(y, z), p(y, fy), p(z, fz), \frac{1}{2s}[p(y, fz) + p(fy, z)]\} = p(y, z).$$

Therefore $p(y, z) < p(y, z)$, which is a contradiction. Therefore $y = z$. Hence f has a unique fixed point. □

Now we state and prove our second main result.

Theorem 3.8. *Let $(X, \leq, p)$ be a complete partially ordered partial b-metric space with $s \geq 1$ and let $\alpha{:}X \times X \to [0, \infty)$ be a function. Let $f : X \to X$ be a self map. Suppose that there exists $\beta \in S$ such that $\alpha(x, y)p(fx, fy) \leq \beta(M(x, y))M(x, y)$ for all $x, y \in X$, where*

$$M(x, y) = max\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2s}[p(x, fy) + p(fx, y)]\}.$$

Assume that

(i) f is α admissible,

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$,

(iii) X is α regular and for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geq 1 \ \ \forall\ n \in N \cup \{0\}$, we have $\alpha(x_m, x_n) \geq 1$ for all $m, n \in N$ with $m < n$,

(iv) $\alpha(x, y) \geq 1\ \forall\ x, y \in Fix(f)$.

Then f has a unique fixed point z in X.

Proof. Let $x_0 \in X$ such that $\alpha(x_0,fx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by $x_n = fx_{n-1}\ \forall\ n \in N$. We have by Theorem 3.7, $\{x_n\}$ is a Cauchy sequence such that $\lim\limits_{n\to\infty} p(x_n, x_{n+1}) = 0$. Therefore $\lim\limits_{n,m\to\infty} p(x_n, x_m)$ exists and is equal to 0. Since $(X, \leq, p)$ is complete, therefore $\{x_n\} \to z$ for some $z \in X$ such that

$$0 = \lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, z) = p(z, z). \tag{3.16}$$

Since X is regular, therefore there exists a sub sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, z) \geq 1\ \forall\ k \in N$. Therefore

$$\begin{aligned} sp(x_{n_k+1}, fz) &\leq \alpha(x_{n_k}, z)sp(fx_{n_k}, fz) \\ &\leq \beta(M(x_{n_k}, z))M(x_{n_k}, z) < M(x_{n_k}, z), \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} M(x_{n_k}, z) &= max\{p(x_{n_k}, z), p(x_{n_k}, fx_{n_k}), p(z, fz), \frac{1}{2s}[p(x_{n_k}, fz) + p(fx_{n_k}, z)]\} \\ &= max\{p(x_{n_k}, z), p(x_{n_k}, x_{n_k+1}), p(z, fz), \frac{1}{2s}[p(x_{n_k}, fz) + p(x_{n_k+1}, z)]\} \\ &\leq max\{p(x_{n_k}, z), p(x_{n_k}, x_{n_k+1}), p(z, fz), \frac{1}{2s}[sp(x_{n_k}, z) + sp(z, fz) - p(z, z) + p(x_{n_k+1}, z)]\} \\ &\leq max\{p(x_{n_k}, z), p(x_{n_k}, x_{n_k+1}), p(z, fz), \frac{1}{2s}[sp(x_{n_k}, z) + sp(z, fz) + p(x_{n_k+1}, z)]\} \\ &= p(z, fz) \text{ for larg } k. \end{aligned} \tag{3.18}$$

Therefore

$$sp(x_{n_k+1}, fz) \leq p(z, fz) \text{ and } \{x_n\} \to z. \tag{3.19}$$

By Lemma 3.5 we have, $\lim\limits_{n\to\infty} sp(x_n, z) = p(z, fz)$. Hence

$$\begin{aligned} p(z, fz) &\leq \beta(p(z, fz))p(z, fz) < p(z, fz) \\ &\Rightarrow p(z, fz) = 0. \end{aligned} \tag{3.20}$$

Therefore $z = fz$ and z is a fixed point of f in X. Assume that u and v, with $u \neq v$ are two fixed points of f.

Then

$$0 < p(u,v) \leq sp(u,v) \leq \alpha(u,v)sp(fu,fv) \leq \beta(M(u,v))M(u,v) < M(u,v),$$

where

$$M(u,v) = max\{p(u,v), p(u,fu), p(v,fv), \frac{1}{2s}[p(u,fv) + p(fu,v)]\} = p(u,v). \tag{3.21}$$

Then

$$0 < p(u,v) \leq \beta(M(u,v))M(u,v) < M(u,v) = p(u,v),$$

which is a contradiction. Therefore, we get $p(u,v) = 0 \Rightarrow u = v$. □

Now we state and prove our third main result.

Theorem 3.9. *Let $(X, \leq, p)$ be a complete partially ordered partial b-metric space with $s \geq 1$ and let $f : X \to X$ be a self map non-decreasing. Let $\alpha : X \times X \to [0,\infty)$ be a function. Suppose that there exists $\beta \in S$ such that*

$$sp(fx,fy) \leq \beta(M(x,y))M(x,y) \text{ for all } x,y \in X \text{ with } x \leq y,$$

where

$$M(x,y) = max\{p(x,y), p(x,fx), p(y,fy), \frac{1}{2s}[p(x,fy) + p(fx,y)]\}.$$

Assume that

(i) there exists $x_0 \in X$ such that $x_0 \leq fx_0$,

(ii) X such that, if a non-decreasing sequence $\{x_n\}$ converges x, then there exists a sub sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \leq x \ \forall \ k \in N$,

(iii) x, y are comparable whenever $x, y \in Fix(f)$.

Then f has a unique fixed point z in X.

Proof. Define mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 \ if x \leq y \\ 0 \ otherwise. \end{cases}$$

$\therefore \alpha(x,y) \geq 1 \Rightarrow x \leq y \Rightarrow fx \leq fy$ (since f is non-decreasing).
$\therefore \alpha(fx,fy) \geq 1$.
$\therefore f$ is α - admissible $\Rightarrow$ condition (i) of Theorem 3.8 holds.
Condition (i) of this theorem $\Rightarrow$ condition (ii) of Theorem 3.8.
Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1 \ \ \forall \ n \in N$ and $x_n \to x \in X \ as \ n \to \infty$. By definition of α, we have $x_n \leq x_{n+1} \ \forall \ n \in N$.
$\therefore \{x_n\}$ is non-decreasing.
$\therefore$ By (ii) of this theorem, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \leq x \ \forall \ k \in N$ and hence X is α-regular. Further, $\alpha(x_m, x_n) \geq 1 \ \ \forall \ m, n \in N$ with $m < n$. Hence (iii) of Theorem 3.8 holds. By condition (iii) of this theorem, $x, y \in Fix(f) \Rightarrow x \leq y \Rightarrow \alpha(x,y) \geq 1 \Rightarrow (iv)$ of Theorem 3.8 holds. Thus hypothesis of Theorem 3.8 holds. Hence by Theorem 3.8 f has a unique fixed point in X. □

Corollary 3.10. *Let $(X, \leq, p)$ be a complete partially ordered partial b-metric space with $s \geq 1$ and let $f : X \to X$ be a self map. Let $\alpha : X \times X \to [0,\infty)$ be a function such that $\alpha(x,y) = 1 \ \forall \ x,y \in X$. Suppose that there exists $\beta \in S$ such that $sp(fx,fy) \leq \beta(M(x,y))M(x,y)$ for all $x,y \in X$, where $M(x,y) = max\{p(x,y), p(x,fx), p(y,fy), \frac{1}{2s}[p(x,fy) + p(fx,y)]\}$. Then f has a unique fixed point z in X*

Now we give an example in support of Corollary 3.10

Example 3.11. Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{10}\}$ with usual ordering. Define

$$p(x,y) = \begin{cases} 0 \ if \ x = y \\ 1 \ if \ x \neq y \in \{0,1\} \\ |x-y| \ if \ x, y \in \{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}\} \\ 4 \ otherwise. \end{cases}$$

Clearly, $(X, \leq, p)$ is a partially ordered partial b-metric space with coefficient $s = \frac{8}{3}$ ([10]). Define $f : X \to X$ by

$$f1 = f\frac{1}{3} = f\frac{1}{5} = f\frac{1}{7} = f\frac{1}{9} = 0 \ ; \ f0 = f\frac{1}{2} = f\frac{1}{4} = f\frac{1}{6} = f\frac{1}{8} = f\frac{1}{10} = \frac{1}{4} \Rightarrow f(X) = \{0, \frac{1}{4}\}$$

and

$$\beta(t) = \begin{cases} \frac{1}{1+t} \ \text{ if } \ t \in (0, \infty), \\ 0 \ \text{ if } t = 0. \end{cases}$$

$\alpha(x,y) = 1 \ \forall \ x, y \in X$.
Let $A = \{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}\}$ and $B = \{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}\} \Rightarrow f(A) = \frac{1}{4}$ and $f(B) = 0$. For $x, y \in X$ the following cases can be observed,

(i) For $x, y \in A \Rightarrow fx = fy = 0 \Rightarrow sp(fx, fy) = 0$,

(ii) For $x, y \in B \Rightarrow fx = fy = \frac{1}{4} \Rightarrow sp(fx, fy) = 0$,

(iii) For $x \in A, y \in B \Rightarrow sp(fx, fy) = (\frac{8}{3})(\frac{1}{4}) = \frac{2}{3}$ where $M(x,y) = 4 \Rightarrow \beta(M(x,y))M(x,y) = \frac{4}{5}$.

Since $f0 = 0$ and $\alpha(0, f0) = 1$. Therefore $0 \in X$ is a fixed point. The hypothesis and conclusions of Corollary 3.10 satisfied.

We observe that Theorems 2.9, 2.10 and 2.11 are true when $s = 1$. Hence theorems 3.5, 3.6 and 4.1 of V. La Rosa et al. [11] are simple corollaries of our main results.

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