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Some common fixed point results in G-cone metric spaces

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Abstract

In this paper we prove some results on points of coincidence and common fixed points for three self-mappings satisfying mappings satisfying various contractive conditions in G-cone metric spaces. Also we deduce some results on common fixed points for two self-mappings satisfying contractive type conditions in G-cone metric spaces.

Keywords: G-cone metric space, common fixed point, coincidence points.
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1. Introduction and preliminaries

Over the past two decades the development of fixed point theory in metric spaces has attracted considerable attention due to numerous applications in areas such as variational and linear inequalities, optimization, and approximation theory. Different generalizations of the notion of a metric space have been proposed by Gähler [7, 8] and by Dhage [5, 6]. However, HA *et al.* [9] have pointed out that the results obtained by Gähler for his 2-metrics are independent, rather than generalizations, of the corresponding results in metric spaces, while in [13] the current authors have pointed out that Dhages notion of a D-metric space is fundamentally flawed and most of the results claimed by Dhage and others are invalid.

In 2005 the concept of generalized metric space was introduced [14]. On the other hand recently Guang and Xian [10] defined the concept of a cone metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. The normality property of cone was an important ingredient in their results (see also, [1], and [2]). Afterward, Rezapour and Hamlbarani [16] omitting the assumption of normality of cone generalized some results of [10].

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A notion of generalized cone metric space and is introduced in [4], and some convergence properties of sequences and some fixed point results are obtained. This space is said to be G-cone metric space.

In this paper we shall study some common fixed points for three self-mappings satisfying mappings satisfying various contractive conditions in G-cone metric spaces. For our study, we need some preliminaries. First we define generalized cone metric space and prove some convergence properties of sequences.

Let E be a real Banach space and let P be a subset of E. P is called a cone if and only if:
(i) P is closed, nonempty, and $P \neq \{0\}$,
(ii) for any $a, b \in [0, \infty)$ and $x, y \in P$, $ax + by \in P$,
(iii) $P \cap (-P) = \{0\}$

Given a cone $P \subset E$, one can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is called normal if there is a number $K > 1$ such that for all $x, y \in E$,

$$0 \leq x \leq y \qquad \text{implies} \qquad \|x\| \leq K\|y\|.$$

The least positive number satisfying the above inequality is called the *normal constant* of P, while $x \ll y$ stands for $y - x \in intP$ (interior of P).
Rezapour and Hamlbarani [16] prove that there are no normal cones with normal constants $K < 1$ and for each $k > 1$ there are cones with normal constants $K > k$.

Definition 1.1 ([4]). Let X be a nonempty set and let P be a cone in real Banach space E. Suppose a mapping $G : X \times X \times X \to E$ satisfies:
(G_1) $G(x, y, z) = 0$ if $x = y = z$,
(G_2) $0 < G(x, x, y)$ whenever $x \neq y$, for all $x, y \in X$,
(G_3) $G(x, x, y) \leq G(x, y, z)$, whenever $y \neq z$,
(G_4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = \cdots$
(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$.
Then G is called a generalized cone metric on X, and X is called a generalized cone metric space or more specifically a G-metric space.

Let X be a G-cone metric space and $\{x_n\}$ be a sequence in X. $\{x_n\}$ is said to be:
(a) Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is $N \in \mathbb{N}$ such that for all $n, m, l > N$, $G(x_n, x_m, x_l) \ll c$.
(b) convergent sequence if for every $c \in E$ with $0 \ll c$, there is $N \in \mathbb{N}$ such that for all $n, m > N, G(x_n, x_m, x) \ll c$, for some fixed $x \in X$. Here x is called the limit of the sequence $\{x_n\}$ and is denote by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(c) A G-cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

A mapping $f : X \to X$ is said to be continuous ot $x_0 \in X$if for any sequence $x_n \to x_0$ we have $f(x_n) \to f(x_0)$.

Lemma 1.2 ([4]). *Let X be a G-cone metric space and $\{x_m\}$, $\{y_n\}$ and $\{z_l\}$ be sequences in X such that $x_m \to x$, $y_n \to y$ and $z_l \to z$, then $G(x_m, y_n, z_l) \to G(x, y, z)$ as $m, n, l \to \infty$.*

A pair (f, T) of self-mappings on X is said to be weakly compatible if they commute of coincidence point (i.e., $fTx = Tfx$ whenever $fx = Tx$). A point $y \in X$ is called point of coincidence of a family T_j,$j \in J$, of self-mappings on X if there exists a point $x \in X$ such that $y = T_jx$, for all $j \in J$.
Suppose S, T and f are three self-mapping on a set X with $S(X) \cup T(X) \subseteq f(X)$. Let x_0 be an arbitrary point of X. Choose a point x_1 in X such that $fx_1 = Sx_0$. This can be done since $S(X) \subseteq f(X)$. Successively, choose a point x_2 in X such that $fx_2 = Tx_1$. Continuing this process having chosen $x_1, \dots, x_{2k}$, we choose x_{2k+1} and x_{2k+2} in X such that

$$fx_{2k+1} = Sx_{2k}, \qquad fx_{2k+2} = Tx_{2k+1}, \qquad k = 0, 1, 2, \dots.$$

The sequence $\{fx_n\}$ is called an S-T-sequence with initial point x_0 (see [3]).

2. Fixed Point Theorems

Lemma 2.1. *Suppose that (X,G) is a G-cone metric space and $S,T,f : X \to X$ are mappings such that $S(X) \cup T(X) \subseteq f(X)$. Also suppose that the following conditions hold:*
(i) For every $x,y,z \in X$, $x \neq y$,

$$G(Sx,Ty,Tz) \leq aG(fx,Sx,Sx) + bG(fy,Ty,Ty) + cG(fz,Tz,Tz) + dG(fx,fy,fz),$$

where a,b,c and d are nonnegative real numbers and $a+b+c+2d < \frac{1}{2}$,
(ii) $G(Sx,Tx,Tx) < G(Sx,fx,fx) + G(fx,Tx,Tx)$, for all $x \in X$, whenever $Sx \neq Tx$.
Then every S-T-sequence with initial point $x_0 \in X$ is a Cauchy sequence.

Proof. Let x_0 be an arbitrary point of X and $\{fx_n\}$ be an S-T-sequence with initial point x_0.
First, we assume that $fx_n \neq fx_{n+1}$, for all $n \in \mathbb{N}$. It implies that $x_n \neq x_{n+1}$, for every $n \in \mathbb{N}$. By condition (i), we have

$$\begin{aligned} G(fx_{2k+1},fx_{2k+2},fx_{2k+2}) &= G(Sx_{2k},Tx_{2k+1},Tx_{2k+1}) \\ &\leq aG(fx_{2k},Sx_{2k},Sx_{2k}) \\ &\quad + (b+c)G(fx_{2k+1},Tx_{2k+1},Tx_{2k+1}) \\ &\quad + dG(fx_{2k},fx_{2k+1},fx_{2k+1}) \\ &= (a+d)G(fx_{2k},fx_{2k+1},fx_{2k+1}) \\ &\quad + (b+c)G(fx_{2k+1},fx_{2k+2},fx_{2k+2}). \end{aligned}$$

Thus

$$G(fx_{2k+1},fx_{2k+2},fx_{2k+2}) \leq \frac{a+d}{1-b-c}G(fx_{2k},fx_{2k+1},fx_{2k+1}). \tag{2.1}$$

Similarly by (G_5) and (i), we obtain

$$\begin{aligned} G(fx_{2k+2},fx_{2k+3},fx_{2k+3}) &= G(Tx_{2k+1},Sx_{2k+2},Sx_{2k+2}) \\ &\leq 2G(Sx_{2k+2},Tx_{2k+1},Tx_{2k+1}) \\ &\leq 2[aG(fx_{2k+2},Sx_{2k+2},Sx_{2k+2}) \\ &\quad + (b+c)G(fx_{2k+1},Tx_{2k+1},Tx_{2k+1}) \\ &\quad + dG(fx_{2k+2},fx_{2k+1},fx_{2k+1})]. \end{aligned}$$

Consequently

$$G(fx_{2k+2},fx_{2k+3},fx_{2k+3}) \leq \frac{2}{1-2a}((b+c)G(fx_{2k+1},fx_{2k+2},fx_{2k+2}) + dG(fx_{2k+2},fx_{2k+1},fx_{2k+1})). \tag{2.2}$$

But by (G_5)

$$\begin{aligned} G(fx_{2k+1},fx_{2k+1},fx_{2k+2}) &= G(Sx_{2k},Sx_{2k},Tx_{2k+1}) \\ &\leq 2G(Sx_{2k},Tx_{2k+1},Tx_{2k+1}) \\ &= 2G(fx_{2k+1},fx_{2k+2},fx_{2k+2}). \end{aligned}$$

Hence by (2.2)

$$G(fx_{2k+2},fx_{2k+3},fx_{2k+3}) \leq \frac{2}{1-2a}(b+c+2d)G(fx_{2k+1},fx_{2k+2},fx_{2k+2}). \tag{2.3}$$

Now, by induction, for each $k = 0,1,2,\ldots$, and letting

$$\lambda = \frac{a+d}{1-b-c} \quad \text{and} \quad \mu = \frac{2(b+c+2d)}{1-2a}, \tag{2.4}$$

we deduce that

$$\begin{aligned}G(fx_{2k+1}, fx_{2k+2}, fx_{2k+2}) &\le \lambda G(fx_{2k}, fx_{2k+1}, fx_{2k+1})\\ &\le \lambda\mu G(fx_{2k-1}, fx_{2k}, fx_{2k})\\ &\le \dots\\ &\le \lambda(\lambda\mu)^k\, G(fx_0, fx_1, fx_1)\end{aligned}$$

and

$$\begin{aligned}G(fx_{2k+2}, fx_{2k+3}, fx_{2k+3}) &\le \mu G(fx_{2k+1}, fx_{2k+2}, fx_{2k+2})\\ &\le \dots \le (\lambda\mu)^{k+1}\, G(fx_0, fx_1, fx_1).\end{aligned}$$

Then $\lambda\mu < 1$, since $a + b + c + 2d < \frac{1}{2}$. Now, for $p < q$, we have

$$\begin{aligned}G(fx_{2p+1}\,, fx_{2q+1}\,, fx_{2q+1}) &\le G(fx_{2p+1}\,, fx_{2p+2}\,, fx_{2p+2})\\ &\quad + G(fx_{2p+2}\,, fx_{2p+3}\,, fx_{2p+3})\\ &\quad + \dots + G(fx_{2q}\,, fx_{2q+1}\,, fx_{2q+1})\\ &\le \left[\lambda \sum_{i=p}^{q-1} (\lambda\mu)^i \;+ \sum_{i=p+1}^{q} (\lambda\mu)^i\right] G(fx_0, fx_1, fx_1)\\ &\le \left[\frac{\lambda(\lambda\mu)^p}{1-\lambda\mu} + \frac{(\lambda\mu)^{p+1}}{1-\lambda\mu}\right] G(fx_0, fx_1, fx_1)\\ &= \lambda(1+\mu)\,\frac{(\lambda\mu)^p}{1-\lambda\mu}\, G(fx_0, fx_1, fx_1)\\ &\le \frac{2(\lambda\mu)^p}{1-\lambda\mu}\, G(fx_0, fx_1, fx_1).\end{aligned}$$

In similar way, we can see that

$$\begin{aligned}G(fx_{2p}\,, fx_{2q+1}\,, fx_{2q+1}) &\le (1+\lambda)\frac{(\lambda\mu)^p}{1-\lambda\mu} G(fx_0, fx_1, fx_1)\\ &\le \frac{2(\lambda\mu)^p}{1-\lambda\mu} G(fx_0, fx_1, fx_1),\\ G(fx_{2p}, fx_{2q}, fx_{2q}) &\le (1+\lambda)\frac{(\lambda\mu)^p}{1-\lambda\mu} G(fx_0, fx_1, fx_1)\\ &\le \frac{2(\lambda\mu)^p}{1-\lambda\mu} G(fx_0, fx_1, fx_1),\\ G(fx_{2p+1}, fx_{2q}, fx_{2q}) &\le \lambda(1+\mu)\frac{(\lambda\mu)^p}{1-\lambda\mu} G(fx_0, fx_1, fx_1)\\ &\le \frac{2(\lambda\mu)^p}{1-\lambda\mu} G(fx_0, fx_1, fx_1).\end{aligned}$$

Hence, for any $0 < n < m$,

$$G(fx_n, fx_m, fx_m) \;\le\; \frac{2(\lambda\mu)^p}{1-\lambda\mu}\, G(fx_0, fx_1, fx_1), \tag{2.5}$$

where p is the integer part of $\frac{n}{2}$.
Fix $0 \ll c$ and choose δ such that $c + N_\delta(0) \subseteq intP$. Since

$$\lim_{p\to\infty} \frac{2(\lambda\mu)^p}{1-\lambda\mu} G(fx_0, fx_1, fx_1) = 0 \tag{2.6}$$

there exists $n_0 \in \mathbb{N}$ such that $\frac{2(\lambda\mu)^p}{1-\lambda\mu} G(fx_0, fx_1, fx_1) \in N_\delta(0)$, for all $p \geq n_0$. Hence,

$$c - \frac{2(\lambda\mu)^p}{1-\lambda\mu} G(fx_0, fx_1, fx_1) \in intP, \tag{2.7}$$

and

$$\frac{2(\lambda\mu)^p}{1-\lambda\mu} G(fx_0, fx_1, fx_1) \ll c. \tag{2.8}$$

Consequently, for all $m, n \in \mathbb{N}$, with $2n_0 < n < m$, we have

$$G(fx_n, fx_m, fx_m) \ll c$$

which implies that $\{fx_n\}$ is a Cauchy sequence. This was for the case that $fx_n = fx_{n+1}$, for any $n \in \mathbb{N}$. Now, suppose that $fx_m = fx_{m+1}$, for some $m \in \mathbb{N}$. If $x_m = x_{m+1}$ and $m = 2k$, by (ii) we have

$$\begin{aligned} G(fx_{2k+1}, fx_{2k+2}, fx_{2k+2}) &= G(Sx_{2k}, Tx_{2k+1}, Tx_{2k+1}) \\ &< G(Sx_{2k}, fx_{2k+1}, fx_{2k+1}) \\ &\quad + G(fx_{2k+1}, Tx_{2k+1}, Tx_{2k+1}) \\ &= G(fx_{2k+1}, fx_{2k+2}, fx_{2k+2}) \end{aligned}$$

which implies $fx_{2k+1} = fx_{2k+2}$ or equivalently $fx_{m+1} = fx_{m+2}$. If $m = 2k+1$, then

$$\begin{aligned} G(fx_{2k+2}, fx_{2k+2}, fx_{2k+3}) &= G(Tx_{2k+1}, Tx_{2k+1}, Sx_{2k+2}) \\ &< G(Sx_{2k+2}, fx_{2k+2}, fx_{2k+2}) \\ &\quad + G(fx_{2k+2}, Tx_{2k+1}, Tx_{2k+1}) \\ &= G(fx_{2k+3}, fx_{2k+2}, fx_{2k+2}). \end{aligned}$$

Hence $fx_{m+1} = fx_{m+2}$.
If $x_m \neq x_{m+1}$, by using (i) we have

$$\begin{aligned} G(fx_{2k+1}, fx_{2k+2}, fx_{2k+2}) &= G(Sx_{2k}, Tx_{2k+1}, Tx_{2k+1}) \\ &\leq aG(fx_{2k}, Sx_{2k}, Sx_{2k}) \\ &\quad + (b+c)G(fx_{2k+1}, Tx_{2k+1}, Tx_{2k+1}) \\ &\quad + dG(fx_{2k}, fx_{2k+1}, fx_{2k+1}) \end{aligned}$$

and so,

$$G(fx_{2k+1}, fx_{2k+2}, fx_{2k+2}) \leq \frac{a}{1-b-c} G(fx_{2k}, fx_{2k+1}, fx_{2k+1}) = 0$$

which implies that $fx_{2k+1} = fx_{2k+2}$, i.e. $fx_{m+1} = fx_{m+2}$. Similarly, we deduce that $fx_{2k+2} = fx_{2k+3}$ and so $fx_n = fx_m$, for every $n \geq m$. Hence $\{fx_n\}$ is a Cauchy sequence. □

The following theorem is a G-cone metric version of Theorem 3.3 of [3].

Theorem 2.2. *Suppose that (X, G) is a G-cone metric space and $S, T, f : X \to X$ are mappings such that $S(X) \cup T(X) \subseteq f(X)$. Also, suppose that the following conditions hold:*
(i) *For every $x, y, z \in X$, $x \neq y$*

$$G(Sx, Ty, Tz) \leq aG(fx, Sx, Sx) + bG(fy, Ty, Ty) + cG(fz, Tz, Tz) + dG(fx, fy, fz),$$

where a, b, c and d are nonnegative real numbers and $a + b + c + 2d < \frac{1}{2}$,
(ii) *$G(Sx, Tx, Tx) < G(Sx, fx, fx) + G(fx, Tx, Tx)$, for all $x \in X$, whenever $Sx \neq Tx$.*

Then
(a) If $f(X)$ or $S(X)\cup T(X)$ is a complete subset of X, then S,T and f have a unique point of coincidence. Moreover, if (S,f) and (T,f) are weakly compatible, then S,T and f have a unique common fixed point.
(b) If (X,G) is complete, (S,f) and (T,f) are weakly compatible and f is continuous or S and T are continuous, then S,T and f have a unique common fixed point.

Proof. Let $x_0\in X$ be an arbitrary point. By previous lemma, every S-T-sequence $\{fx_n\}$ with initial point x_0 is a Cauchy sequence. If $f(X)$ is a complete subset of X, there exist $u,v\in X$ such that $fx_n\to v=fu$ (this holds also if $S(X)\cup T(X)$ is complete with $v\in T(X)$).
From

$$\begin{aligned}G(fu,Tu,Tu)&\le G(fu,fx_{2n+1},fx_{2n+1})+G(fx_{2n+1},Tu,Tu)\\&\le G(fu,fx_{2n+1},fx_{2n+1})+aG(fx_{2n},Sx_{2n},Sx_{2n})\\&\quad+(b+c)G(fu,Tu,Tu)+dG(fx_{2n},fu,fu),\end{aligned}$$

we obtain

$$G(fu,Tu,Tu)\le\frac{1}{1-b-c}\Big(G(fu,fx_{2n+1},fx_{2n+1})+aG(fx_{2n},fx_{2n+1},fx_{2n+1})+dG(fx_{2n},fu,fu)\Big). \tag{2.9}$$

Fix $0\ll\alpha$ and choose $n_0\in\mathbb{N}$ be such that

$$\begin{gathered}G(fu,fx_{2n+1},fx_{2n+1})\ll\beta\cdot\alpha,\\G(fx_{2n},fx_{2n+1},fx_{2n+1})\ll\beta\cdot\alpha,\quad G(fx_{2n},fu,fu))\ll\beta\cdot\alpha,\end{gathered}$$

for all $n\ge n_0$, where $\beta=\frac{1-b-c}{1+a+d}$. This is possible, since $fx_n\to f(u)$. Consequently $G(fu,Tu,Tu)\ll\alpha$ and hence $G(fu,Tu,Tu)\ll\frac{\alpha}{m}$, for every $m\in\mathbb{N}$. It means that $\frac{\alpha}{m}-G(fu,Tu,Tu)\in intP$. As $m\to\infty$, we have $-G(fu,Tu,Tu)\in P$ and so $G(fu,Tu,Tu)=0$. This implies that $fu=Tu=v$. Also from

$$\begin{aligned}G(fu,Su,Su)&\le G(fu,fx_{2n+2},fx_{2n+2})+G(fx_{2n+2},Su,Su)\\&\le G(fu,fx_{2n+2},fx_{2n+2})+2G(Su,Tx_{2n+1},Tx_{2n+1})\\&\le G(fu,fx_{2n+2},fx_{2n+2})+2aG(fu,Su,Su)\\&\quad+2(b+c)G(fx_{2n+1},Tx_{2n+1},Tx_{2n+1})\\&\quad+2dG(fu,fx_{2n+1},fx_{2n+1}),\end{aligned}$$

we obtain

$$\begin{aligned}G(fu,Su,Su)\le\frac{1}{1-2a}\Big(&G(fu,fx_{2n+2},fx_{2n+2})\\&+2(b+c)G(fx_{2n+1},Tx_{2n+1},Tx_{2n+1})\\&+2dG(fu,fx_{2n+1},fx_{2n+1})\Big).\end{aligned}$$

Fix $0\ll\eta$ and choose $n_1\in\mathbb{N}$ be such that

$$G(fu,fx_{2n+2},fx_{2n+2})\ll\gamma\cdot\eta,\qquad G(fx_{2n+1},fx_{2n+2},fx_{2n+2})\ll\gamma\cdot\eta,$$

and $G(fu,fx_{2n+1},fx_{2n+1}))\ll\gamma\cdot\eta$, for all $n\ge n_1$, where $\gamma=\frac{1-a}{1+b+c+d}$. Consequently $G(fu,Su,Su)\ll\eta$ and hence $G(fu,Su,Su)\ll\frac{\eta}{m}$, for every $m\in\mathbb{N}$. It means that $\frac{\eta}{m}-G(fu,Su,Su)\in intP$. As $m\to\infty$, we have $-G(fu,Su,Su)\in P$ and so $G(fu,Su,Su)=0$. This implies that $fu=Su=Tu=v$.
Now, we show that the point of coincidence of S,T and f is unique. For this, assume that there exist $u^*,v^*\in X$ such that $fu^*=Su^*=Tu^*=v^*$. Then

$$G(v,v^*,v^*)=\qquad G(Su,Tu^*,Tu^*)\le aG(fu,Su,Su)+(b+c)G(fu^*,Tu^*,Tu^*)+dG(fu,fu^*,fu^*).$$

So

$$G(v, v^*, v^*) \le dG(v, v^*, v^*)$$

and $d < 1$ implies that $v = v^*$.
Moreover, if (S, f) and (T, f) are weakly compatible, then

$$Sv = Sfu = fSu = fv \text{ and } Tv = Tfu = fTu = fv.$$

It implies that $fv = Sv = Tv = w$(say). But the point of coincidence is unique, so $v = w$ and v is a unique common fixed point of S, T and f.

(b) Let $x_0 \in X$ be an arbitrary point. By lemma 2.1, every S-T-sequence $\{fx_n\}$ with initial point x_0 is a Cauchy sequence. Since X is complete, there is a $y \in X$ such that $y_n = fx_n \to y$. First, suppose that f is continuous. Then,

$$f^2x_n \to fy, \quad fTx_n \to fy, \quad fSx_n \to fy. \tag{2.10}$$

But (f, T) and (f, S) are weakly compatible, so $Tfx_n \to fy$ and $Sfx_n \to fy$. By lemma 1.2 and (2.10) we have

$$G(fy, Ty, Ty) = \lim_{n\to\infty} G(fSx_n, Ty, Ty) = \lim_{n\to\infty} G(Sfx_n, Ty, Ty). \tag{2.11}$$

But, by assumption (i), we have

$$G(Sfx_n, Ty, Ty) \le aG(f^2x_n, Sfx_n, Sfx_n) + (b + c)G(fy, Ty, Ty) + dG(f^2x_n, fy, fy). \tag{2.12}$$

When $n \to \infty$

$$G(fy, Ty, Ty) \le (b + c)G(fy, Ty, Ty), \tag{2.13}$$

so $G(fy, Ty, Ty) = 0$ and $fy = Ty$. Similarly, one can see that $Ty = y$ and so $fy = Ty = y$. Moreover, $Sy = y$. Indeed

$$G(Sy, y, y) = \lim_{n\to\infty} G(Sy, Tx_n, Tx_n) \tag{2.14}$$

and by assumption (i)

$$G(Sy, Tx_n, Tx_n) \le aG(fy, Sy, Sy) + (b + c)G(fx_n, Tx_n, Tx_n) + dG(fy, fx_n, fx_n) \tag{2.15}$$

When $n \to \infty$

$$G(Sy, y, y) \le aG(y, Sy, Sy) \le 2aG(Sy, y, y) \tag{2.16}$$

hence $Sy = y$, since $a \le a + b + c + 2d < \frac{1}{2}$. So $Sy = Ty = fy = y$ and y is a common fixed point of f, S and T. Now we show that y is unique. For this, suppose that there exists another point $y^* \in X$ such that $fy^* = Ty^* = Sy^* = y^*$. We have

$$G(y, y^*, y^*) = G(Sy, Ty^*, Ty^*) \le aG(fy, Sy, Sy) + (b + c)G(fy^*, Ty^*, Ty^*) + dG(fy, fy^*, fy^*)$$

then

$$G(y, y^*, y^*) \le dG(y, y^*, y^*) \tag{2.17}$$

and so $y = y^*$. Hence, if f is continuous, then f, T and S have a unique common fixed point.
In the case that S and T are continuous, by using the same argument of the previous case, we have

$$T^2x_n \to Ty, \quad Tfx_n \to Ty$$

and

$$S^2x_n \to Sy, \quad Sfx_n \to Sy.$$

We show that $Sy = Ty$.

$$G(Sy, Ty, Ty) = \lim_{n\to\infty} G(S^2x_n, T^2x_n, T^2x_n) \tag{2.18}$$

By (i), we have

$$\begin{aligned} G(S^2x_n, T^2x_n, T^2x_n) \leq\ & aG(fSx_n, S^2x_n, S^2x_n) \\ & + (b+c)G(fTx_n, T^2x_n, T^2x_n) \\ & + dG(fSx_n, fTx_n, fTx_n) \end{aligned} \tag{2.19}$$

Taking the limit $n \to \infty$, we deduce

$$G(Sy, Ty, Ty) \leq dG(Sy, Ty, Ty). \tag{2.20}$$

So $G(Sy, Ty, Ty) = 0$ and $Sy = Ty$. But, $Ty = y$, since

$$G(y, Ty, Ty) = \lim_{n\to\infty} G(Sx_n, T^2x_n, T^2x_n), \tag{2.21}$$

also

$$\begin{aligned} G(Sx_n, T^2x_n, T^2x_n) \leq\ & aG(fx_n, Sx_n, Sx_n) \\ & + (b+c)G(fTx_n, T^2x_n, T^2x_n) \\ & + dG(fx_n, fTx_n, fTx_n), \end{aligned} \tag{2.22}$$

which yields

$$G(y, Ty, Ty) \leq dG(y, Ty, Ty). \tag{2.23}$$

So $Ty = y$ and $Sy = Ty = y$. Now the fact that, $S(X) \cup T(X) \subseteq f(X)$, implies that there exists $y\prime \in X$ such that $y = Sy = Ty = fy\prime$. Hence

$$G(Sy, Ty\prime, Ty\prime) = \lim_{n\to\infty} G(S^2x_n, Ty\prime, Ty\prime). \tag{2.24}$$

Moreover,

$$G(S^2x_n, Ty\prime, Ty\prime) \leq aG(fSx_n, S^2x_n, S^2x_n) + (b+c)G(fy\prime, Ty\prime, Ty\prime) + dG(fSx_n, fy\prime, fy\prime). \tag{2.25}$$

letting $n \to \infty$, we have

$$G(Sy, Ty\prime, Ty\prime) \leq (b+c)G(Sy, Ty\prime, Ty\prime). \tag{2.26}$$

So $y = Ty = Sy = Ty\prime$. But (f, T) is weakly compatible, so

$$fy = fTy\prime = Tfy\prime = Ty = y.$$

Hence, y is a common fixed point of f, T and S. The proof of uniqueness of y is similar to the proof of uniqueness in part (a). □

If we choose $S = T$ in Theorem 2.2, (a), we deduce one part of the following theorem. Also this theorem a G-cone metric version of Theorem 3.4 of [3].

Theorem 2.3. *Let (X, G) be a G-cone metric space and let $T, f : X \to X$ be such that $T(X) \subset f(X)$. Assume that the following condition holds:*

$$G(Tx, Ty, Tz) \leq aG(fx, Tx, Tx) + bG(fy, Ty, Ty) + cG(fz, Tz, Tz) + dG(fx, fy, fz), \tag{2.27}$$

for all $x, y, z \in X$ where a,b,c and d are nonnegative real numbers and $a+b+c+2d < \frac{1}{2}$ or $a+b+c+d < 1$. If $f(X)$ or $T(X)$ is a complete subset of X, then T and f have a unique point of coincidence. Moreover, if (T, f) is weakly compatible, then T and f have a unique common fixed point.

Proof. For the case that $a+b+c+2d<\frac{1}{2}$, it is enough to put $S=T$ in Theorem 2.2, (a).

Let $a+b+c+d<1$. If $f(X)$ is a complete subset of X, then there exist $u,v\in X$ such that $y_n\to v=fu$ (this holds also if $T(X)$ is complete with $v\in T(X)$).
From

$$\begin{aligned}G(fu,Tu,Tu)&\le G(fu,Tx_n,Tx_n)+G(Tx_n,Tu,Tu)\\&\le G(fu,Tx_n,Tx_n)+aG(fx_n,Tx_n,Tx_n)\\&\quad+(b+c)G(fu,Tu,Tu)+dG(fx_n,fu,fu),\end{aligned}$$

we obtain

$$G(fu,Tu,Tu)\le\frac{1}{1-b-c}\big(G(fu,y_n,y_n)+aG(fx_n,Tx_n,Tx_n)+dG(y_{n-1},fu,fu)\big).\tag{2.28}$$

But $y_n\to v=fu$, so for every $\alpha\in E$, $0\ll\alpha$, there exists $n_0\in\mathbb{N}$ such that

$$G(y_{n-1},fu,fu)\ll\left(\frac{1-b-c}{1+d}\right)\alpha\qquad\text{and}\qquad G(y_n,y_n,fu)\ll\left(\frac{1-b-c}{1+d}\right)\alpha,\tag{2.29}$$

for every $n\ge n_0$. Thus

$$G(fu,Tu,Tu)\ll\frac{1}{1-b-c}\left((\frac{1-b-c}{1+d})\alpha+d(\frac{1-b-c}{1+d})\alpha\right)=\alpha\tag{2.30}$$

Hence, for all $m\ge1$, $G(fu,Tu,Tu)\ll\frac{\alpha}{m}$ and so, $\frac{\alpha}{m}-G(fu,Tu,Tu)\in intP$. But $\frac{\alpha}{m}\to0$ as $m\to\infty$, therefore $-G(fu,Tu,Tu)\in P$ and it means that $fu=Tu=v$. Hence, v is a point of coincidence of f and T. We show that v is unique. For this, suppose that there exist $u^*,v^*\in X$ such that $fu^*=Tu^*=v^*$. From

$$\begin{aligned}G(v,v^*,v^*)=G(Tu,Tu^*,Tu^*)&\le aG(fu,Tu,Tu)+(b+c)G(fu^*,Tu^*,Tu^*)+dG(fu,fu^*,fu^*)\\&=dG(v,v^*,v^*),\end{aligned}$$

we obtain $v=v^*$. Moreover, if (f,T) is weakly compatible, then

$$Tv=Tfu=fTu=fv=w.$$

But, the point of coincidence of f and T is a unique point v, then $w=v$ and $Tv=fv=v$. So, T and f have a unique common fixed point. □

If we choose $S=T$ in Theorem 2.2 (b), we deduce one part of the following theorem.

Theorem 2.4. *Let (X,G) be a complete G-cone metric space and let $T,f:X\to X$ be such that $T(X)\subset f(X)$. Assume that the following condition holds:*

$$G(Tx,Ty,Tz)\le aG(fx,Tx,Tx)+bG(fy,Ty,Ty)+cG(fz,Tz,Tz)+dG(fx,fy,fz),\tag{2.31}$$

for all $x,y,z\in X$ where a,b,c and d are nonnegative real numbers and $a+b+c+2d<\frac{1}{2}$ or $a+b+c+d<1$. If f or T is continuous and (T,f) is weakly compatible, then T and f have a unique common fixed point.

Proof. For the case that $a+b+c+2d<\frac{1}{2}$, it is enough to put $S=T$ in Theorem 2.2, (b).

Let $a+b+c+d<1$ and $x_0\in X$ be arbitrary. There exists $x_1\in X$ such that $Tx_1=fx_0$, since, $T(X)\subset f(X)$,. Successively, there exists $x_2\in X$ such that $Tx_1=fx_2$. Continuing this process having chosen $x_1,x_2,\dots,x_n$ in X we may choose $x_{n+1}\in X$ such that $y_n:=Tx_n=fx_{n+1}$. We have

$$\begin{aligned}G(y_n,y_{n+1},y_{n+1})&=G(Tx_n,Tx_{n+1},Tx_{n+1})\\&\le aG(fx_n,Tx_n,Tx_n)+(b+c)G(fx_{n+1},Tx_{n+1},Tx_{n+1})+dG(fx_n,fx_{n+1},fx_{n+1})\end{aligned}$$

and so,

$$G(y_n, y_{n+1}, y_{n+1}) \leq \left(\frac{a+d}{1-b-c}\right) G(y_{n-1}, y_n, y_n) \leq \ldots \leq q^n G(y_0, y_1, y_1)$$

where $q = \frac{a+d}{1-b-c}$. Trivially $0 \leq q < 1$, since $0 \leq a+b+c+d < 1$.
Hence, for every $n, m \in \mathbb{N}$, $n < m$

$$\begin{aligned} G(y_n, y_m, y_m) &\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) \\ &\quad + \ldots + G(y_{m-1}, y_m, y_m) \\ &\leq \left(q^n + q^{n+1} + \ldots + q^{m-1}\right) G(y_0, y_1, y_1) \\ &\leq \left(\frac{q^n}{1-q}\right) G(y_0, y_1, y_1). \end{aligned}$$

Let $0 \ll c$ be given. Choose δ such that $c + N_\delta(0) \subseteq intP$, where $N_\delta(0) = \{y \in E : \|y\| < \delta\}$. Also, choose a natural number N_1 such that $\frac{q^n}{1-q} G(y_0, y_1, y_1) \in N_\delta(0)$, for all $n \geq N_1$. $c - \frac{q^n}{1-q} G(y_0, y_1, y_1) \in intP$ and $\frac{q^n}{1-q} G(y_0, y_1, y_1) \ll c$, for all $n \geq N_1$. So we have $G(y_n, y_m, y_m) \ll c$, for all $m > n$. Thus $\{y_n\}$ is a Cauchy sequence.
Since, X is complete, there exists $y \in X$ such that $y_n = Tx_n = fx_{n+1} \to y$. Now suppose that f is continuous. Then,

$$f^2 x_n \to fy, \quad fTx_n \to fy. \tag{2.32}$$

But (f, T) is weakly compatible, so $Tfx_n \to fy$. By lemma 1.2 and (2.32) we have

$$G(fy, Ty, Ty) = \lim_{n\to\infty} G(fTx_n, Ty, Ty) = \lim_{n\to\infty} G(Tfx_n, Ty, Ty). \tag{2.33}$$

But, by assumption, we have

$$G(Tfx_n, Ty, Ty) \leq aG(f^2x_n, Tfx_n, Tfx_n) + (b+c)G(fy, Ty, Ty) + dG(f^2x_n, fy, fy). \tag{2.34}$$

So as $n \to \infty$, we get

$$G(fy, Ty, Ty) \leq (b+c)G(fy, Ty, Ty), \tag{2.35}$$

which implies that $G(fy, Ty, Ty) = 0$ and $fy = Ty$.
Also $Ty = y$. Indeed

$$G(Ty, y, y) = \lim_{n\to\infty} G(Ty, Tx_n, Tx_n). \tag{2.36}$$

On the other hand by our assumption

$$G(Ty, Tx_n, Tx_n) \leq aG(fy, Ty, Ty) + (b+c)G(fx_n, Tx_n, Tx_n) + dG(fy, fx_n, fx_n). \tag{2.37}$$

Hence

$$G(Ty, y, y) \leq aG(Ty, y, y). \tag{2.38}$$

which yields $G(Ty, y, y) = 0$ and $fy = Ty = y$.
Now suppose that T is continuous. We have

$$T^2 x_n \to Ty, \quad Tfx_n \to Ty. \tag{2.39}$$

As a same argument of first part of the proof, we have

$$G(Ty, y, y) = \lim_{n\to\infty} G(T^2x_n, Tx_n, Tx_n) \tag{2.40}$$

but

$$G(T^2x_n, Tx_n, Tx_n) \leq aG(fTx_n, T^2x_n, T^2x_n) + (b+c)G(fx_n, Tx_n, Tx_n) + dG(fTx_n, fx_n, fx_n). \tag{2.41}$$

Taking the limit $n \to \infty$, we have

$$G(Ty, y, y) \leq dG(Ty, y, y) \tag{2.42}$$

and so, $Ty = y$. Moreover, since $T(X) \subset f(X)$, there exists $y\prime \in X$ such that $y = Ty = fy\prime$. Similarly

$$G(Ty, Ty\prime, Ty\prime) = \lim_{n\to\infty} G(T^2 x_n, Ty\prime, Ty\prime) \tag{2.43}$$

and by assumption

$$G(T^2x_n, Ty\prime, Ty\prime) \leq aG(fTx_n, T^2x_n, T^2x_n) + (b+c)G(fy\prime, Ty\prime, Ty\prime) + dG(fTx_n, fy\prime, fy\prime) \tag{2.44}$$

taking the limit $n \to \infty$, we have

$$G(Ty, Ty\prime, Ty\prime) \leq (b+c)G(fy\prime, Ty\prime, Ty\prime) \tag{2.45}$$

and so, $y = Ty = Ty\prime$. Now, since (T, f) is weakly compatible, we deduce

$$fy = fTy\prime = Tfy\prime = Ty = y.$$

Hence $fy = Ty = y$. Similarly, we can see that y is a unique common fixed point of f and T. □

The following corollary is a G-cone metric version of Corollary 3.7 [3] and Theorem 2.6 of [16]. Let $a = b$ and $c = d = 0$ so by Theorem 2.3 we have

Corollary 2.5. *Let (X, G) be a G-cone metric space and let $T, f : X \to X$ be such that $T(X) \subset f(X)$. Assume that the following condition holds:*

$$G(Tx, Ty, Ty) \leq a(G(fx, Tx, Tx) + G(fy, Ty, Ty)) \tag{2.46}$$

for all $x, y, z \in X$ where $0 \leq a < \frac{1}{2}$.
If $f(X)$ or $T(X)$ is a complete subset of X, then T and f have a unique point of conicidence. Moreover,if (T, f) is weakly compatible, then T and f have a unique common fixed point.

A similar conclusion can be made by using Theorem 2.4, for the case that f or T is continuous.

By letting $f = I$, $a = b$ and $c = d = 0$ in Theorem 2.4, we get the following corollary.

Corollary 2.6. *Let (X, G) be a complete G-cone metric space and let $T : X \to X$ be such that, for all $x, y, z \in X$,*

$$G(Tx, Ty, Ty) \leq a(G(x, Tx, Tx) + G(y, Ty, Ty)) \tag{2.47}$$

where $0 \leq a < \frac{1}{2}$. Then T has a unique fixed point.

Proof. From previous theorem, if we choose $f = I$, we deduce this Corollary. □

Similarly with $a = b = c = 0$ in Theorem 2.3 we have

Corollary 2.7. *Let (X, G) be a G-cone metric space and let $T, f : X \to X$ be such that $T(X) \subset f(X)$. Assume that the following condition holds:*

$$G(Tx, Ty, Tz) \leq \lambda G(fx, fy, fz), \tag{2.48}$$

for all $x, y, z \in X$ where $0 \leq \lambda < 1$.
If $f(X)$ or $T(X)$ is a complete subset of X, then T and f have a unique point of coincidence. Moreover, if (T, f) is weakly compatible, then T and f have a unique common fixed point.

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