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Quasi-contractive mappings and their common fixed points on G-cone metric spaces

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Abstract

In this paper we prove some results on points of coincidence and common fixed points for two self-mappings satisfying various quasi-contractive conditions in G-cone metric spaces.

Keywords: Quasi-contractive mapping, G-cone metric space, common fixed point, coincidence points.
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1. Introduction and preliminaries

Different generalizations of the notion of a metric space have been proposed by Gähler [7, 8] and by Dhage [5, 6]. However, Ha *et al.* [9] have pointed out that the results obtained by Gähler for his 2-metrics are independent, rather than generalizations, of the corresponding results in metric spaces, while in [13] the current authors have pointed out that Dhages notion of a D-metric space is fundamentally flawed and most of the results claimed by Dhage and others are invalid.

In 2005 the concept of generalized metric space was introduced [14]. On the other hand recently Guang and Xian [10] defined the concept of a cone metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. The normality property of cone was an important ingredient in their results (see also, [1], and [2]). Afterward, Rezapour and Hamlbarani [16] omitting the assumption of normality of cone generalized some results of [10].

A notion of generalized cone metric space and is introduced in [4], and some convergence properties of sequences and some fixed point results are obtained. This space is said to be G-cone metric space. For some common fixed point results on cone metric and generalized cone metric spaces on may see [3], [11] and [12].

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In this paper we shall study common fixed points for two self-mappings satisfying various quasi-contractive conditions in G-cone metric spaces. For our study, we need some preliminaries. First we state the concept of generalized cone metric space.

Let E be a real Banach space and let P be a subset of E. P is called a cone if and only if
(i) P is closed, nonempty, and $P \neq \{0\}$,
(ii) for any $a, b \in [0, \infty)$ and $x, y \in P$, $ax + by \in P$,
(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, one can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is called normal if there is a number $K > 1$ such that for all $x, y \in E$,

$$0 \leq x \leq y \quad \text{implies} \quad \|x\| \leq K\|y\|.$$

The least positive number satisfying the above inequality is called the *normal constant* of P, while $x \ll y$ stands for $y - x \in intP$ (interior of P).
Rezapour and Hamlbarani [16] prove that there are no normal cones with normal constants $K < 1$ and for each $k > 1$ there are cones with normal constants $K > k$.

Definition 1.1 ([4]). Let X be a nonempty set and let P be a cone in real Banach space E. Suppose a mapping $G : X \times X \times X \to E$ satisfies
(G_1) $G(x, y, z) = 0$ if $x = y = z$,
(G_2) $0 < G(x, x, y)$ whenever $x \neq y$, for all $x, y \in X$,
(G_3) $G(x, x, y) \leq G(x, y, z)$, whenever $y \neq z$,
(G_4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = \cdots$
(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$.
Then G is called a generalized cone metric on X, and X is called a generalized cone metric space or more specifically a G-cone metric space.

Let X be a G-cone metric space and $\{x_n\}$ be a sequence in X. $\{x_n\}$ is said to be a
(a) Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is $N \in \mathbb{N}$ such that for all $n, m, l > N$, $G(x_n, x_m, x_l) \ll c$.
(b) convergent sequence if for every $c \in E$ with $0 \ll c$, there is $N \in \mathbb{N}$ such that for all $n, m > N, G(x_n, x_m, x) \ll c$, for some fixed $x \in X$. Here x is called the limit of $\{x_n\}$ and is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(c) A G-cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for any sequence $x_n \to x_0$, we have $f(x_n) \to f(x_0)$.

Lemma 1.2 ([4]). *Let X be a G-cone metric space and $\{x_m\}$, $\{y_n\}$ and $\{z_l\}$ be sequences in X such that $x_m \to x$, $y_n \to y$ and $z_l \to z$, then $G(x_m, y_n, z_l) \to G(x, y, z)$ as $m, n, l \to \infty$.*

A point $y \in X$ is called point of coincidence of a family $T_j, j \in J$, of self-mappings on X if there exists a point $x \in X$ such that $y = T_j x$, for all $j \in J$.
A pair (f, T) of self-mappings on X is said to be weakly compatible if they commute on their coincidence point (i.e., $fTx = Tfx$ whenever $fx = Tx$).

2. Fixed Point Theorems

In this section some common fixed point theorems for two self-mapping, with quasi-contractive conditions will be obtained in G-cone metric spaces which are extensions of some results in [15]. Our results will be proved without normality condition on the cone.

The part (a) of the following theorem is a generalization of Theorem 2.1 of [15] for two self mapping and in G-cone metric case.

Theorem 2.1. *Let (X, G) be a G-cone metric space and let $T, f : X \to X$ be a weakly compatible pair of functions such that $T(X) \subset f(X)$. Assume that for some $\lambda \in [0, \frac{1}{2})$ and for all $x, y, z \in X$, there exists*

$$\omega \in \{G(fx, fy, fz), G(fx, Tx, Tx), G(fy, Ty, Ty), G(fz, Tz, Tz), G(fx, Ty, Ty), G(fy, Tz, Tz), G(fz, Tx, Tx)\}$$

such that

$$G(Tx, Ty, Tz) \le \lambda\omega. \tag{2.1}$$

(a) If $f(X)$ or $T(X)$ is a complete subspace of X, then T and f have a unique common fixed point.
(b) If f or T is continuous, then T and f have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. There exists $x_1 \in X$ such that $Tx_0 = fx_1$, since $T(X) \subset f(X)$. Successively, there exists $x_2 \in X$ such that $Tx_1 = fx_2$. Continuing this process, having chosen $x_1, x_2, \dots, x_n$ in X, we may choose x_{n+1} such that $y_n := Tx_n = fx_{n+1}$.
We are going to find a point of coincidence of f and T. If there exists $n \in \mathbb{N}$ such that $Tx_{n+1} = y_{n+1} = y_n = Tx_n = fx_{n+1}$ then x_{n+1} is a point of coincidence of f and T, so without lose of generality, we may assume that $y_n \ne y_{n+1}$, for all $n \in \mathbb{N}$. We know

$$G(y_n, y_{n+1}, y_{n+1}) = G(Tx_n, Tx_{n+1}, Tx_{n+1}) \le \lambda\omega, \tag{2.2}$$

for some

$$\begin{aligned}\omega \in \{&G(fx_n, fx_{n+1}, fx_{n+1}), G(fx_n, Tx_n, Tx_n), G(fx_{n+1}, Tx_{n+1}, Tx_{n+1}),\\ &G(fx_n, Tx_{n+1}, Tx_{n+1}), G(fx_{n+1}, Tx_n, Tx_n)\}\\ = \{&G(y_{n-1}, y_n, y_n), G(y_n, y_{n+1}, y_{n+1}), G(y_{n-1}, y_{n+1}, y_{n+1})\}.\end{aligned}$$

ω can not be $G(y_n, y_{n+1}, y_{n+1})$, since if $\omega = G(y_n, y_{n+1}, y_{n+1})$, then

$$G(y_n, y_{n+1}, y_{n+1}) \le \lambda.G(y_n, y_{n+1}, y_{n+1})$$

or equivalently

$$(\lambda - 1)G(y_n, y_{n+1}, y_{n+1}) \in P.$$

But $(1 - \lambda)G(y_n, y_{n+1}, y_{n+1}) \in P$, since $0 \le \lambda < \frac{1}{2}$ and P is a cone. Hence $G(y_n, y_{n+1}, y_{n+1}) = 0$ and $y_n = y_{n+1}$, which is a contradiction.
If $\omega = G(y_{n-1}, y_{n+1}, y_{n+1})$, then by (G_5), we have

$$G(y_{n-1}, y_{n+1}, y_{n+1}) \le G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1}).$$

Hence $G(y_n, y_{n+1}, y_{n+1}) \le \lambda\omega$, for some

$$\omega \in \{G(y_{n-1}, y_n, y_n), G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1})\}.$$

Without lose of generality, we may assume that $\omega = G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1})$, since if $\omega = G(y_{n-1}, y_n, y_n)$, then

$$G(y_n, y_{n+1}, y_{n+1}) \le \lambda G(y_{n-1}, y_n, y_n) \le \lambda(G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1})).$$

Therefore

$$G(y_n, y_{n+1}, y_{n+1}) \le \lambda(G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1}))$$

and so

$$G(y_n, y_{n+1}, y_{n+1}) \le \frac{\lambda}{1-\lambda} G(y_{n-1}, y_n, y_n) = qG(y_{n-1}, y_n, y_n), \tag{2.3}$$

where $q = \frac{\lambda}{1-\lambda}$. Trivially $0 \leq q < 1$. Using a similar argument we get

$$G(y_n, y_{n+1}, y_{n+1}) \leq qG(y_{n-1}, y_n, y_n) \leq \ldots \leq q^n G(y_0, y_1, y_1). \tag{2.4}$$

Now, for all $m, n \in \mathbb{N}$, $n < m$, we have

$$\begin{aligned} G(y_n, y_m, y_m) &\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + \ldots + G(y_{m-1}, y_m, y_m) \\ &\leq (q^n + \ldots + q^{m-1})G(y_0, y_1, y_1) \\ &\leq \frac{q^n}{1-q} G(y_0, y_1, y_1). \end{aligned}$$

Let $0 \ll c$ be given. Choose δ such that $c + N_\delta(0) \subseteq intP$, where $N_\delta(0) = \{y \in E : \|y\| < \delta\}$. Also, choose a natural number N_1 such that $\frac{q^n}{1-q} G(y_0, y_1, y_1) \in N_\delta(0)$, for all $n \geq N_1$. Thus $c - \frac{q^n}{1-q} G(y_0, y_1, y_1) \in intP$ and $\frac{q^n}{1-q} G(y_0, y_1, y_1) \ll c$, for all $n \geq N_1$. So we have $G(y_n, y_m, y_m) \ll c$, for all $m > n$. Hence $\{y_n\}$ is a Cauchy sequence.

For proof of (a), if $f(X)$ is a complete subset of X, then there exist $u, v \in X$ such that $y_n \to v = fu$ (this holds also if $T(X)$ is complete with $v \in T(X)$).

From

$$G(fu, Tu, Tu) \leq G(fu, Tx_n, Tx_n) + G(Tx_n, Tu, Tu) \leq G(fu, Tx_n, Tx_n) + \lambda\omega,$$

where

$$\omega \in \{G(fx_n, fu, fu), G(fx_n, Tx_n, Tx_n), G(fu, Tu, Tu), G(fx_n, Tu, Tu), G(fu, Tx_n, Tx_n)\},$$

we obtain

$$\begin{aligned} G(fu, Tu, Tu) \leq G(fu, Tx_n, Tx_n) + \lambda(&G(fx_n, fu, fu) + G(fx_n, Tx_n, Tx_n) \\ &+ G(fu, Tu, Tu) + G(fx_n, Tu, Tu) + G(fu, Tx_n, Tx_n)). \end{aligned}$$

Letting $n \to \infty$ and using Lemma 1.2, we have

$$G(fu, Tu, Tu) \leq 2\lambda G(fu, Tu, Tu),$$

which implies that $G(fu, Tu, Tu) = 0$ or $Tu = fu = v$. Hence v is a point of coincidence of f and T.

To prove uniqueness, assume that there exist $u^*, v^* \in X$ such that $fu^* = Tu^* = v^*$. From

$$G(v, v^*, v^*) = G(Tu, Tu^*, Tu^*) \leq \lambda\omega,$$

where

$$\omega \in \{G(fu, fu^*, fu^*), G(fu, Tu^*, Tu^*), G(fu^*, Tu, Tu)\},$$

there exists $\omega \in \{G(v, v^*, v^*), G(v^*, v, v)\}$ such that $G(v, v^*, v^*) \leq \lambda\omega$. So, if $\omega = G(v, v^*, v^*)$ then $v = v^*$, since $0 \leq \lambda < \frac{1}{2}$. Also if $\omega = G(v^*, v, v)$, by using a similar process, we have $G(v^*, v, v) \leq \lambda G(v, v^*, v^*)$ and so,

$$G(v, v^*, v^*) \leq \lambda G(v^*, v, v) \leq \lambda^2 G(v, v^*, v^*).$$

Hence $v = v^*$ and f and T have a unique point of coincidence.

Now, if (f, T) is weakly compatible, then

$$Tv = Tfu = fTu = fv,$$

which implies that $Tv = fv = \alpha$ (say). Thus α is a point of coincidence of f and T therefore, $v = \alpha$. Hence v is a unique common fixed point of f and T.

(b) We saw that the sequence $y_n = Tx_n = fx_{n+1}$ is a Cauchy sequence. Since X is a complete space, there exists $y \in X$ such that $y_n = Tx_n = fx_{n+1} \to y$. First suppose that f is continuous. Then

$$f^2x_n \to fy, \quad fTx_n \to fy. \tag{2.5}$$

But the pair (f, T) is weakly compatible, so $Tfx_n \to fy$. By Lemma 1.2 we have

$$G(fy, Ty, Ty) = \lim_{n\to\infty} G(fTx_n, Ty, Ty) = \lim_{n\to\infty} G(Tfx_n, Ty, Ty). \tag{2.6}$$

Also for any $n \in \mathbb{N}$, there exists

$$s_n \in \{G(f^2x_n, fy, fy), G(f^2x_n, Tfx_n, Tfx_n), G(f^2x_n, Ty, Ty), G(fy, Ty, Ty), G(fy, Tfx_n, Tfx_n)\}$$

such that

$$G(Tfx_n, Ty, Ty) \le \lambda s_n.$$

Hence

$$\begin{aligned} G(Tfx_n, Ty, Ty) \le \lambda(&G(f^2x_n, fy, fy) + G(f^2x_n, Tfx_n, Tfx_n) + G(f^2x_n, Ty, Ty) \\ &+ G(fy, Ty, Ty) + G(fy, Tfx_n, Tfx_n)). \end{aligned} \tag{2.7}$$

Letting $n \to \infty$, we have

$$G(fy, Ty, Ty) \le 2\lambda G(fy, Ty, Ty).$$

Thus $G(fy, Ty, Ty) = 0$, since $0 \le \lambda < \frac{1}{2}$, and so $fy = Ty$. Moreover,

$$G(Ty, y, y) = \lim_{n\to\infty} G(Ty, Tx_n, Tx_n) \tag{2.8}$$

and for any $n \in \mathbb{N}$, there is

$$s_n \in \{G(fy, fx_n, fx_n), G(fx_n, Tx_n, Tx_n), G(fy, Ty, Ty), G(fx_n, Ty, Ty), G(fy, Tx_n, Tx_n)\}$$

such that

$$G(Ty, Tx_n, Tx_n) \le \lambda s_n, \tag{2.9}$$

We shall consider constant subsequences $\{s_{n,i}\}$, $i = 1, 2, 3, 4$ of the sequence $\{s_n\}$ with elements of the form $G(fy, fx_n, fx_n)$, $G(fx_n, Tx_n, Tx_n)$, $G(fy, Tx_n, Tx_n)$ and $G(fx_n, Ty, Ty)$, respectively. It is clear that

$$\lim_{n\to\infty} s_{n,2} = G(y, y, y) = 0.$$

Also $\lim_{n\to\infty} s_{n,i} = G(Ty, y, y)$, $i = 1, 3$ and $\lim_{n\to\infty} s_{n,4} = G(y, Ty, Ty)$. Hence, by (2.8) and (2.9), we have $G(Ty, y, y) \le 0$ or

$$G(Ty, y, y) \le \lambda G(Ty, y, y)$$

or

$$G(Ty, y, y) \le \lambda G(y, Ty, Ty) \le \lambda^2 G(Ty, y, y).$$

In any case it implies that $G(Ty, y, y) = 0$ and so $Ty = y$, since $0 \le \lambda < \frac{1}{2}$. Hence $fy = Ty = y$ and y is a common fixed point of f and T.

Now, assume that T is continuous. Then

$$T^2x_n \to Ty, \quad Tfx_n \to Ty.$$

We have

$$G(Ty, y, y) = \lim_{n\to\infty} G(T^2x_n, Tx_n, Tx_n). \tag{2.10}$$

Furthermore, for each $n \in \mathbb{N}$, there exists

$$\begin{aligned} s_n \in & \{G(fTx_n, fx_n, fx_n), G(fTx_n, T^2x_n, T^2x_n), G(fx_n, Tx_n, Tx_n), \\ & G(fTx_n, Tx_n, Tx_n), G(fx_n, T^2x_n, T^2x_n)\}, \end{aligned}$$

such that

$$G(T^2x_n, Tx_n, Tx_n) \leq \lambda s_n.$$

We shall consider the constant subsequences $\{s_{n,i}\}$, $i = 1, 2, \dots, 5$, of the sequence $\{s_n\}$ with elements of the form $G(fTx_n, fx_n, fx_n)$, $G(fTx_n, T^2x_n, T^2x_n)$, $G(fx_n, Tx_n, Tx_n)$, $G(fTx_n, Tx_n, Tx_n)$ and $G(fx_n, T^2x_n, T^2x_n)$, respectively. By using a same argument used in the previous part, we deduce $Ty = y$.
We know $T(X) \subset f(X)$, so there exists $y_0 \in X$ such that $y = Ty = fy_0$. But $y = Ty = Ty_0$. Indeed we have

$$G(Ty, Ty_0, Ty_0) = \lim_{n\to\infty} G(T^2x_n, Ty_0, Ty_0). \tag{2.11}$$

Also, for each $n \in \mathbb{N}$, there exists

$$s_n \in \{G(fTx_n, fy_0, fy_0), G(fTx_n, T^2x_n, T^2x_n), G(fy_0, Ty_0, Ty_0), G(fy_0, T^2x_n, T^2x_n), G(fTx_n, Ty_0, Ty_0)\},$$

such that

$$G(T^2x_n, Ty_0, Ty_0) \leq \lambda s_n.$$

Consequently,

$$\begin{aligned} G(T^2x_n, Ty_0, Ty_0) \leq \lambda(& G(fTx_n, fy_0, fy_0) + G(fTx_n, T^2x_n, T^2x_n) \\ & + G(fy_0, Ty_0, Ty_0) + G(fy_0, T^2x_n, T^2x_n) + G(fTx_n, Ty_0, Ty_0)). \end{aligned}$$

Letting $n \to \infty$, we have

$$G(Ty, Ty_0, Ty_0) \leq 2\lambda G(Ty, Ty_0, Ty_0).$$

Thus $G(Ty, Ty_0, Ty_0) = 0$, since $0 \leq \lambda < \frac{1}{2}$, and so $y = Ty = Ty_0$. Weakly compatibility of (T, f) implies that

$$fy = fTy_0 = Tfy_0 = Ty = y.$$

Hence $fy = Ty = y$.

Now, we show that y is the unique common fixed point of f and T. For this, if $y^* \in X$ be another common fixed point of f and T, then

$$G(y, y^*, y^*) = G(Ty, Ty^*, Ty^*) \leq \lambda\omega,$$

for some

$$\omega \in \{G(fy, fy^*, fy^*), G(fy, Ty, Ty), G(fy^*, Ty^*, Ty^*), G(fy, Ty^*, Ty^*), G(fy^*, Ty, Ty)\}.$$

It means that, there exists $\omega \in \{G(y, y^*, y^*), G(y^*, y, y)\}$ such that

$$G(y, y^*, y^*) \leq \lambda\omega.$$

If $\omega = G(y, y^*, y^*)$, the fact that $0 \leq \lambda < \frac{1}{2}$, implies that $y = y^*$. Also, if $\omega = G(y^*, y, y)$, by following the previous process we have

$$G(y, y^*, y^*) \leq \lambda G(y^*, y, y) \leq \lambda^2 G(y, y^*, y^*),$$

which implies that $y = y^*$.

□

Corollary 2.2. *Let (X, G) be a complete G-cone metric space and let $T : X \to X$ be such that for all $x, y, z \in X$, there exists*

$$\omega \in \{G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz), G(x,Ty,Ty), G(y,Tz,Tz), G(z,Tx,Tx)\}, \quad (2.12)$$

such that $G(Tx,Ty,Tz) \leq \lambda\omega$, where $\lambda \in [0, \frac{1}{2})$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Proof. If we choose $f = I$, in part (a) of Theorem 2.1 we deduce the first part. We show that T is G-continuous at u. For this, assume that $\{y_n\}$ is a sequence in X such that $\lim_{n\to\infty} y_n = u$, then $G(Ty_n, Tu, Ty_n) \leq \lambda\omega$, for some

$$\omega \in \{G(y_n,u,y_n), G(y_n,Ty_n,Ty_n), G(y_n,Tu,Tu), G(u,Ty_n,Ty_n)\},$$

which implies that

$$G(Ty_n, Tu, Ty_n) \leq \lambda\omega, \quad (2.13)$$

where

$$\omega \in \{G(y_n,u,y_n), G(y_n,Ty_n,Ty_n), G(y_n,u,u)\}.$$

But by (G_5), we have

$$G(y_n,Ty_n,Ty_n) \leq G(y_n,u,u) + G(u,Ty_n,Ty_n).$$

Hence using (2.13), we obtain one of the following cases,

1. $G(Ty_n,Tu,Ty_n) \leq \lambda G(y_n,y_n,u)$,
2. $G(Ty_n,Tu,Ty_n) \leq \lambda G(y_n,u,y_n)$,
3. $G(Ty_n,Tu,Ty_n) \leq qG(y_n,u,y_n)$,

where $q = \frac{\lambda}{1-\lambda}$. In each case take the limit as $n \to \infty$ to see that $G(Ty_n,u,Ty_n) \to 0$ and so, $\{Ty_n\}$ is convergent to $u = Tu$. Therefore T is G-continuous at u. □

Corollary 2.3. *Let (X,G) be a complete G-cone metric space and let $T : X \to X$ be a mapping such that for some $m \in N$ and for all $x,y,z \in X$, there exists*

$$\begin{aligned}\omega \in &\{G(x,y,z), G(x,T^m x,T^m x), G(y,T^m y,T^m y), G(z,T^m z,T^m z),\\ &G(x,T^m y,T^m y), G(y,T^m z,T^m z), G(z,T^m x,T^m x)\}\end{aligned}$$

such that

$$G(T^m x, T^m y, T^m z) \leq \lambda\omega \quad (2.14)$$

where $\lambda \in [0, \frac{1}{2})$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Proof. From Corollary 2.2, T^m has a unique fixed point (say u), that is $T^m(u) = u$. But $T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u))$, so $T(u)$ is another fixed point for T^m and by uniqueness $Tu = u$. □

Theorem 2.4. *Let (X,G) be a G-cone metric space and let $T,f : X \to X$ be such that $T(X) \subset f(X)$. Assume that for all $x,y,z \in X$, there exists*

$$\omega \in \{G(fx,Ty,Ty) + G(fy,Tx,Tx), G(fy,Tz,Tz) + G(fz,Ty,Ty), G(fx,Tz,Tz) + G(fz,Tx,Tx)\},$$

such that

$$G(Tx,Ty,Tz) \leq \lambda\omega \quad (2.15)$$

where $\lambda \in [0, \frac{1}{2})$.
(a) If $f(X)$ or $T(X)$ is a complete subset of X, then T and f have a unique point of coincidence. Moreover, if (T,f) is weakly compatible, then T and f have a unique common fixed point.
(b) If X is complete and f or T is continuous, then T and f have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. By a similar argument as used in Theorem 2.1, we may find sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n = Tx_n = fx_{n+1}$. Also, we may assume that $y_n \neq y_{n+1}$, for all $n \in \mathbb{N}$, otherwise, there exists $n \in \mathbb{N}$ such that $Tx_{n+1} = y_{n+1} = y_n = Tx_n = fx_{n+1}$ and so x_{n+1} is a point of coincidence of f and T. We have

$$G(y_n, y_{n+1}, y_{n+1}) = G(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq \lambda\omega, \tag{2.16}$$

for some

$$\begin{aligned}\omega \in &\{G(fx_n, Tx_{n+1}, Tx_{n+1}) + G(fx_{n+1}, Tx_n, Tx_n), G(fx_{n+1}, Tx_{n+1}, Tx_{n+1}) \\ &+ G(fx_{n+1}, Tx_{n+1}, Tx_{n+1})\}.\end{aligned}$$

Equivalently there exists

$$\omega \in \{G(y_{n-1}, y_{n+1}, y_{n+1}), 2G(y_n, y_{n+1}, y_{n+1})\},$$

such that

$$G(y_n, y_{n+1}, y_{n+1}) \leq \lambda\omega. \tag{2.17}$$

ω can not be $2G(y_n, y_{n+1}, y_{n+1})$, since if $\omega = 2G(y_n, y_{n+1}, y_{n+1})$, then from (2.17), $(2\lambda-1)G(y_n, y_{n+1}, y_{n+1}) \in P$. But $(1-2\lambda)G(y_n, y_{n+1}, y_{n+1}) \in P$, since $0 \leq \lambda < \frac{1}{2}$ and P is a cone. Hence $G(y_n, y_{n+1}, y_{n+1}) = 0$ and so $y_n = y_{n+1}$ which is a contradiction. Thus $\omega = G(y_{n-1}, y_{n+1}, y_{n+1})$. By (G_5), we have

$$G(y_{n-1}, y_{n+1}, y_{n+1}) \leq G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1}).$$

Hence

$$G(y_n, y_{n+1}, y_{n+1}) \leq \lambda G(y_{n-1}, y_{n+1}, y_{n+1}) \leq \lambda(G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1})), \tag{2.18}$$

and so

$$G(y_n, y_{n+1}, y_{n+1}) \leq \frac{\lambda}{1-\lambda} G(y_{n-1}, y_n, y_n) = qG(y_{n-1}, y_n, y_n). \tag{2.19}$$

where $q = \frac{\lambda}{1-\lambda}$. Trivially $0 \leq q < 1$ and by using (2.19), we have

$$\begin{aligned}G(y_n, y_m, y_m) &\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + \ldots + G(y_{m-1}, y_m, y_m) \\ &\leq (q^n + \ldots + q^{m-1})G(y_0, y_1, y_1) \\ &\leq \frac{q^n}{1-q} G(y_0, y_1, y_1),\end{aligned}$$

for all $m, n \in \mathbb{N}$, $n < m$. Let $0 \ll c$ be given. Choose δ such that $c + N_\delta(0) \subseteq P$, where $N_\delta(0) = \{y \in E : \|y\| < \delta\}$. Also choose a natural number N_1 such that $\frac{q^n}{1-q} G(y_0, y_1, y_1) \in N_\delta(0)$, for all $n \geq N_1$. Then $c - \frac{q^n}{1-q} G(y_0, y_1, y_1) \in intP$ and $\frac{q^n}{1-q} G(y_0, y_1, y_1) \ll c$, for all $n \geq N_1$. So we have $G(y_n, y_m, y_m) \ll c$, for all $m > n$. Thus $\{y_n\}$ is a Cauchy sequence.

Proof of (a). If $f(X)$ is a complete subspace of X, then there exist $u, v \in X$ such that $y_n \to v = fu$ (this holds also if $T(X)$ is complete with $v \in T(X)$). From

$$G(fu, Tu, Tu) \leq G(fu, Tx_n, Tx_n) + G(Tx_n, Tu, Tu) \leq G(fu, Tx_n, Tx_n) + \lambda\omega,$$

where

$$\omega \in \{2G(fu, Tu, Tu), G(fx_n, Tu, Tu) + G(fu, Tx_n, Tx_n)\},$$

we obtain one of the following cases

1. $G(fu, Tu, Tu) \leq G(fu, Tx_n, Tx_n) + \lambda(G(fx_n, Tu, Tu) + G(fu, Tx_n, Tx_n))$
2. $G(fu, Tu, Tu) \leq G(fu, Tx_n, Tx_n) + 2\lambda G(fu, Tu, Tu)$

In each case, take the limit as $n \to \infty$ to see that

1. $G(fu, Tu, Tu) \leq \lambda G(fu, Tu, Tu)$
2. $G(fu, Tu, Tu) \leq 2\lambda G(fu, Tu, Tu)$,

respectively. Thus $fu = Tu = v$, since $0 \leq \lambda < \frac{1}{2}$.
To prove uniqueness of u, assume that there exist $u^*, v^* \in X$ such that $fu^* = Tu^* = v^*$. From

$$G(v, v^*, v^*) = G(Tu, Tu^*, Tu^*) \leq \lambda\omega,$$

with

$$\omega \in \{G(fu, Tu^*, Tu^*) + G(fu^*, Tu, Tu), G(fu^*, Tu^*, Tu^*) + G(fu^*, Tu^*, Tu^*)\},$$

we obtain

$$G(v, v^*, v^*) \leq \lambda(G(v, v^*, v^*) + G(v^*, v, v)), \tag{2.20}$$

which implies that

$$G(v, v^*, v^*) \leq \frac{\lambda}{1-\lambda} G(v^*, v, v). \tag{2.21}$$

Similarly,

$$G(v^*, v, v) \leq \frac{\lambda}{1-\lambda} G(v, v^*, v^*). \tag{2.22}$$

So,

$$G(v, v^*, v^*) \leq q^2 G(v, v^*, v^*),$$

where $q = \frac{\lambda}{1-\lambda}$. Hence $v = v^*$, since $0 \leq q < 1$, and f and T have a unique point of coincidence. Moreover, if (f, T) is weakly compatible, then

$$Tv = Tfu = fTu = fv,$$

which implies $Tv = fv = \alpha$ (say). Then α is a point of coincidence of f and T therefore, $v = \alpha$.Thus v is a unique common fixed point of f and T.

(b) We saw that the sequence $y_n = Tx_n = fx_{n+1}$ is a Cauchy sequence. By completeness of X, there exists $y \in X$ such that $y_n = Tx_n = fx_{n+1} \to y$.
First suppose that f is continuous. Then

$$f^2 x_n \to fy, \quad fTx_n \to fy. \tag{2.23}$$

But (f, T) is weakly compatible, so $Tfx_n \to fy$. By Lemma 1.2 we have

$$G(fy, Ty, Ty) = \lim_{n\to\infty} G(fTx_n, Ty, Ty) = \lim_{n\to\infty} G(Tfx_n, Ty, Ty). \tag{2.24}$$

Also, there exists

$$s_n \in \{G(f^2 x_n, Ty, Ty) + G(fy, Tfx_n, Tfx_n), 2G(fy, Ty, Ty)\},$$

such that

$$G(Tfx_n, Ty, Ty) \leq \lambda s_n.$$

It is evident that the elements of the sequence $\{s_n\}$ are of the form $G(f^2 x_n, Ty, Ty) + G(fy, Tfx_n, Tfx_n)$ or $2G(fy, Ty, Ty)$. We shall consider subsequences $\{s_{n,i}\}$, $i = 1, 2$ of the sequence $\{s_n\}$, such that all the elements of the sequence $\{s_{n,i}\}$, $i = 1, 2$, are of the form $G(f^2 x_n, Ty, Ty) + G(fy, Tfx_n, Tfx_n)$ and $2G(fy, Ty, Ty)$, respectively. It is clear that $\lim_{n\to\infty} s_{n,1} = G(fy, Ty, Ty)$, and $\lim_{n\to\infty} s_{n,2} = 2G(fy, Ty, Ty)$. Hence, by (2.24), we have

$$G(fy, Ty, Ty) \leq \lambda G(fy, Ty, Ty)$$

or

$$G(fy, Ty, Ty) \leq 2\lambda G(fy, Ty, Ty),$$

which implies that $G(fy, Ty, Ty) = 0$ and so $Ty = fy$, since $0 \leq \lambda < \frac{1}{2}$.
Now in contrary, suppose that $Ty \neq y$, then

$$G(Ty, y, y) = \lim_{n\to\infty} G(Ty, Tx_n, Tx_n). \tag{2.25}$$

But, there exists

$$\omega \in \{G(fy, Tx_n, Tx_n) + G(fx_n, Ty, Ty), 2G(fx_n, Tx_n, Tx_n)\}$$

such that

$$G(Ty, Tx_n, Tx_n) \leq \lambda\omega.$$

Thus

1. $G(Ty, Tx_n, Tx_n) \leq \lambda(G(fy, Tx_n, Tx_n) + G(fx_n, Ty, Ty))$, or
2. $G(Ty, Tx_n, Tx_n) \leq 2\lambda G(fx_n, Tx_n, Tx_n)$.

In each case take the limit as $n \to \infty$ to see that

1. $G(Ty, y, y) \leq \lambda(G(Ty, y, y) + G(y, Ty, Ty))$,
2. $G(Ty, y, y) \leq 2\lambda G(y, y, y) = 0$.

If (2) holds then by (G_2), we have $Ty = y$ which is a contradiction. So (1) is valid, which implies that

$$G(Ty, y, y) \leq \left(\frac{\lambda}{1-\lambda}\right) G(y, Ty, Ty).$$

Similarly,

$$G(y, Ty, Ty) \leq \left(\frac{\lambda}{1-\lambda}\right) G(Ty, y, y),$$

and so

$$G(Ty, y, y) \leq \left(\frac{\lambda}{1-\lambda}\right)^2 G(Ty, y, y).$$

Hence $fy = Ty = y$.
Now, suppose that T is continuous. Then

$$T^2x_n \to Ty, \quad Tfx_n \to Ty. \tag{2.26}$$

As a similar argument to the first part of the proof, we have

$$G(Ty, y, y) = \lim_{n\to\infty} G(T^2x_n, Tx_n, Tx_n), \tag{2.27}$$

but

$$G(T^2x_n, Tx_n, Tx_n) \leq \lambda\omega, \tag{2.28}$$

where

$$\omega \in \{G(fTx_n, Tx_n, Tx_n) + G(fx_n, T^2x_n, T^2x_n), 2G(fx_n, Tx_n, Tx_n)\}.$$

Thus one of following cases is valid

1. $G(T^2x_n, Tx_n, Tx_n) \leq \lambda(G(fTx_n, Tx_n, Tx_n) + G(fx_n, T^2x_n, T^2x_n)$
2. $G(T^2x_n, Tx_n, Tx_n) \leq 2\lambda G(fx_n, Tx_n, Tx_n)$

In each case, letting $n \to \infty$ we get

1. $G(Ty, y, y) \leq \lambda(G(Ty, y, y) + G(y, Ty, Ty))$

2. $G(Ty, y, y) \leq 2\lambda G(y, y, y) = 0$.

which implies that $Ty = y$, similar to the proof when f is continuous.
We are going to show that $fy = y$. By the fact that $T(X) \subset f(X)$, there exists $y_0 \in X$ such that $y = Ty = fy_0$. We have

$$G(Ty, Ty_0, Ty_0) = \lim_{n\to\infty} G(T^2 x_n, Ty_0, Ty_0) \tag{2.29}$$

and

$$G(T^2 x_n, Ty_0, Ty_0) \leq \lambda\omega,$$

where

$$\omega \in \{G(fTx_n, Ty_0, Ty_0) + G(fy_0, T^2 x_n, T^2 x_n), 2G(fy_0, Ty_0, Ty_0)\}.$$

One of the following cases may be occurred

1. $G(T^2 x_n, Ty_0, Ty_0) \leq \lambda(G(fTx_n, Ty_0, Ty_0) + G(fy_0, T^2 x_n, T^2 x_n)$
2. $G(T^2 x_n, Ty_0, Ty_0) \leq 2\lambda G(fy_0, Ty_0, Ty_0)$.

In any case taking the limit as $n \to \infty$, we have

1. $G(Ty, Ty_0, Ty_0) \leq \lambda G(Ty, Ty_0, Ty_0)$
2. $G(Ty, Ty_0, Ty_0) \leq 2\lambda G(Ty, Ty_0, Ty_0)$.

This implies that $G(Ty, Ty_0, Ty_0) = 0$, since $\lambda \in [0, \frac{1}{2})$, so $y = Ty = Ty_0$. Hence, if (f, T) is weakly compatible, then

$$fy = fTy_0 = Tfy_0 = Ty = y.$$

Now, if y^* is another point of X such that $fy^* = Ty^* = y^*$, then we have

$$G(y^*, y, y) \leq \lambda\omega, \tag{2.30}$$

for some

$$\omega \in \{G(fy^*, Ty, Ty) + G(fy, Ty^*, Ty^*), G(fy, Ty, Ty) + G(fy, Ty, Ty)\}.$$

So

$$G(y^*, y, y) \leq \lambda(G(y^*, y, y) + G(y, y^*, y^*))$$

and

$$G(y^*, y, y) \leq \left(\frac{\lambda}{1-\lambda}\right) G(y, y^*, y^*).$$

Similarly $G(y, y^*, y^*) \leq \left(\frac{\lambda}{1-\lambda}\right) G(y^*, y, y)$. Thus

$$G(y^*, y, y) \leq \left(\frac{\lambda}{1-\lambda}\right)^2 G(y^*, y, y).$$

Hence, $y = y^*$. □

If in Theorem 2.4 we put the identity mapping instead of f, then we obtain the following corollaries.

Corollary 2.5. *Let (X, G) be a complete G-cone metric space and let $T : X \to X$ be such that for all $x, y, z \in X$ there exists*

$$\omega \in \{G(x, Ty, Ty) + G(y, Tx, Tx), G(y, Tz, Tz) + G(z, Ty, Ty), G(x, Tz, Tz) + G(z, Tx, Tx)\},$$

such that $G(Tx, Ty, Tz) \leq \lambda\omega$, where $\lambda \in [0, \frac{1}{2})$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Corollary 2.6. *Let (X,G) be a complete G-cone metric space and let $T : X \to X$ be a mapping such that for some $m \in N$ and for all $x, y, z \in X$, there exists*

$$\begin{aligned}\omega \in &\{G(x,T^m y,T^m y)+G(y,T^m x,T^m x), G(y,T^m z,T^m z)+G(z,T^m y,T^m y),\\ &G(x,T^m z,T^m z)+G(z,T^m x,T^m x)\},\end{aligned}$$

such that

$$G(Tx,Ty,Tz) \leq \lambda\omega,$$

where $\lambda \in [0, \frac{1}{2})$. Then T has a unique fixed point (say u) and T is G-continuous at u.

Theorem 2.7. *Let (X,G) be a G-cone metric space and let $T, f : X \to X$ be such that $T(X) \subset f(X)$. Assume that for all $x, y, z \in X$, there exists*

$$\omega \in \{G(fy,Ty,Ty)+G(fx,Ty,Ty), 2G(fy,Tx,Tx)\},$$

such that

$$G(Tx,Ty,Tz) \leq \lambda\omega, \tag{2.31}$$

for some $\lambda \in [0, \frac{1}{3})$. If $f(X)$ or $T(X)$ is a complete subset of X, then T and f have a unique point of coincidence. Moreover, if (T, f) is weakly compatible, then T and f have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Using a similar argument used in Theorem 1.2, we obtain sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n = Tx_n = fx_{n+1}$. By hypotheses

$$G(y_n,y_{n+1},y_{n+1}) = G(Tx_n,Tx_{n+1},Tx_{n+1}) \leq \lambda\omega, \tag{2.32}$$

for some

$$\omega \in \{G(fx_{n+1},Tx_{n+1},Tx_{n+1})+G(fx_n,Tx_{n+1},Tx_{n+1}), 2G(fx_{n+1},Tx_n,Tx_n)\}.$$

This, by definition of y_n, implies that

$$\omega = G(y_n,y_{n+1},y_{n+1})+G(y_{n-1},y_{n+1},y_{n+1}).$$

Hence, by (2.32) and (G_5), we have

$$\begin{aligned}G(y_n,y_{n+1},y_{n+1}) &\leq \lambda(G(y_n,y_{n+1},y_{n+1})+G(y_{n-1},y_{n+1},y_{n+1}))\\ &\leq \lambda(G(y_n,y_{n+1},y_{n+1})+G(y_{n-1},y_n,y_n)+G(y_n,y_{n+1},y_{n+1}))\\ &= \lambda(2G(y_n,y_{n+1},y_{n+1})+G(y_{n-1},y_n,y_n)).\end{aligned}$$

Consequently,

$$G(y_n,y_{n+1},y_{n+1}) \leq \frac{\lambda}{1-2\lambda}G(y_{n-1},y_n,y_n) = qG(y_{n-1},y_n,y_n), \tag{2.33}$$

where $q = \frac{\lambda}{1-2\lambda}$. Trivially $0 \leq q < 1$ and by using (2.33) successively, we obtain

$$G(y_n,y_{n+1},y_{n+1}) \leq qG(y_{n-1},y_n,y_n) \leq \ldots \leq q^n G(y_0,y_1,y_1).$$

Thus, for all $m, n \in \mathbb{N}$, $n < m$,

$$\begin{aligned}G(y_n,y_m,y_m) &\leq G(y_n,y_{n+1},y_{n+1})+G(y_{n+1},y_{n+2},y_{n+2})+\ldots+G(y_{m-1},y_m,y_m)\\ &\leq (q^n+\ldots+q^{m-1})G(y_0,y_1,y_1)\\ &\leq \frac{q^n}{1-q}G(y_0,y_1,y_1).\end{aligned}$$

Fix $0 \ll c$ and let δ be such that $c+N_\delta(0) \subseteq P$. Also, choose a natural number N_1 such that $\frac{q^n}{1-q}G(y_0, y_1, y_1) \in N_\delta(0)$, for all $n \geq N_1$. Then $c - \frac{q^n}{1-q}G(y_0, y_1, y_1) \in intP$ and $\frac{q^n}{1-q}G(y_0, y_1, y_1) \ll c$, for all $n \geq N_1$. So we have $G(y_n, y_m, y_m) \ll c$, for all $m > n$. Thus $\{y_n\}$ is a Cauchy sequence.
If $f(X)$ is a complete subspace of X, there exist $u, v \in X$ such that $y_n \to v = fu$ (this holds also if $T(X)$ is complete with $v \in T(X)$).
From

$$G(fu, Tu, Tu) \leq G(fu, Tx_n, Tx_n) + G(Tx_n, Tu, Tu) \leq G(fu, Tx_n, Tx_n) + \lambda\omega$$

where

$$\omega \in \{G(fu, Tu, Tu) + G(fx_n, Tu, Tu), 2G(fu, Tx_n, Tx_n)\},$$

we obtain the following cases

1. $G(fu, Tu, Tu) \leq G(fu, Tx_n, Tx_n) + \lambda(G(fu, Tu, Tu) + G(fx_n, Tu, Tu))$
2. $G(fu, Tu, Tu) \leq G(fu, Tx_n, Tx_n) + 2\lambda G(fu, Tx_n, Tx_n)$

In each case taking the limit as $n \to \infty$, we may see that

1. $G(fu, Tu, Tu) \leq 2\lambda G(fu, Tu, Tu)$
2. $G(fu, Tu, Tu) \leq 0$.

Thus $fu = Tu = v$, since $0 \leq \lambda < \frac{1}{3}$.
To prove uniqueness, assume that there exist $u^*, v^* \in X$ such that $fu^* = Tu^* = v^*$. From

$$G(v, v^*, v^*) = G(Tu, Tu^*, Tu^*) \leq \lambda\omega,$$

where

$$\omega \in \{G(fu^*, Tu^*, Tu^*) + G(fu, Tu^*, Tu^*), 2G(fu^*, Tu, Tu)\},$$

we obtain

$$G(v, v^*, v^*) \leq \lambda\omega \tag{2.34}$$

for some

$$\omega \in \{G(v, v^*, v^*), 2G(v^*, v, v)\}.$$

This implies that,

$$G(v, v^*, v^*) \leq \lambda G(v, v^*, v^*), \tag{2.35}$$

or

$$G(v, v^*, v^*) \leq 2\lambda G(v^*, v, v) \leq 4\lambda^2 G(v, v^*, v^*). \tag{2.36}$$

In each case, $v = v^*$ and f and T have a unique point of coincidence. Moreover, if (f, T) is weakly compatible, then

$$Tv = Tfu = fTu = fv,$$

which implies $Tv = fv = \alpha$ (say). Thus α is a point of coincidence of f and T. Therefore, $v = \alpha$. Hence v is a unique common fixed point of f and T. □

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