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Some results on the homogeneous space G/H

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Abstract

Let G be a locally compact group and H be a compact subgroup of G. Every G-invariant measure on G/H induces a convolution on the Banach space $L^p(G/H,\mu)$, $p \geq 1$. In this article, we extend the Young's inequality and some other results from $L^p(G)$ to $L^p(G/H,\mu)$.

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1. Introduction

Throughout the paper, let G be a locally compact group and H be a subgroup of it by left Haar measures λ and ξ respectively. For measurable functions f and g on G the convolution is defined as follow

$$f * g(x) = \int_G f(y)g(y^{-1}x)d\lambda(y) \tag{1.1}$$

provided that the function $y \mapsto f(y)g(y^{-1}x)$ is λ-integrable. The convolution $f * g$ exists if $(f * g)(x)$ is defined for almost all $x \in G$.

Transitive G-spaces and locally compact groups are similar in many respects and so are important objects in harmonic analysis which are well known as homogeneous spaces. We consider most of these objects as quotient spaces G/H where H is some closed subgroup of G, see Proposition 2.44 of [2]. Let Δ_G (resp. Δ_H) be the modular function of G (resp. H). A ρ-function for the pair (G,H) is a continuous function $\rho : G \longrightarrow (0,\infty)$ which satisfies $\rho(xh) = \Delta_H(h)\Delta_G(h)^{-1}\rho(x)$ for each $x \in G$ and $h \in H$. If H is compact,

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then the existence of an invariant measure on G/H, which is called G-invariant measure, is guaranteed [2]. In [3], Ghaani Farashahi by using the surjective linear map $T_H : C_c(G) \longrightarrow C_c(G/H)$ defined by

$$T_H(f)(xH) = \int_H \frac{f(xh)}{\rho(xh)} d\xi(h), \quad (x \in G, f \in C_c(G)) \tag{1.2}$$

introduces a well-defined convolution on $C_c(G/H)$. According to this definition, for functions φ and ψ in $C_c(G/H)$ we have

$$\varphi * \psi = T_H(\varphi_\pi * \psi_\pi). \tag{1.3}$$

φ_π (resp. ψ_π) in (1.3) refers to $\rho\varphi o\pi$ (resp. $\rho\psi o\pi$) where ρ is a ρ-function for the pair (G, H) and $\pi : G \longrightarrow G/H$ is the surjective canonical mapping defined by $\pi(x) = xH$ for all $x \in G$. The collection of all φ_π where $\varphi \in C_c(G/H)$ and $\varphi_\pi(xh) = \varphi_\pi(x)$ for all $x \in G$ and $h \in H$, denoted by $C_c(G : H)$, is a sub-algebra of $L^1(G)$. If H is compact, then T_H is a norm decreasing map in L^p-norm and therefore has a unique extension to a bounded linear map from $L^p(G)$ to $L^p(G/H)$ by the continuity for all $p \geq 1$. This extends the convolution from $L^p(G)$ to $L^p(G/H)$, see [3, 5, 6] for more details.

This paper is devoted to Young's inequality and some other results in homogeneous spaces. From here on, we assume that H is compact subgroup of G and μ is a G-invariant measure on G/H. Also we denote the functional space $L^p(G/H, \mu)$ by $L^p(G/H)$.

2. Main results

Let H be a closed subgroup of a locally group G and μ a G-invariant measure on G/H. First of all, note that the surjective map $T_H : C_c(G) \longrightarrow C_c(G/H)$ defined by (1.2) satisfies the Mackey-Bruhat formula,

$$\int_{G/H} T_H(f)(xH) d\mu(xH) = \int_G f(x) d\lambda(x) \tag{2.1}$$

which is also well known as the Weil's formula. Also if H is compact, then we can extend this map to a norm decreasing operator from $L^p(G)$ to $L^p(G/H)$, see also Lemma 2.2 of [1] for more details. To state our result we need the following two lemmas;

Lemma 2.1 ([2, 4])**.** *Let r,p,q be real numbers and $1 < p, q < \infty$. Let $f \in L^p(G)$ and $g \in L^q(G)$. We have*

(i) *If $\dfrac{1}{p} + \dfrac{1}{q} > 1$ and $\dfrac{1}{p} + \dfrac{1}{q} - \dfrac{1}{r} = 1$, then $\|f * g\|_{L^r(G)} \leq \|f\|_{L^p(G)} \|g\|_{L^q(G)}$.*

(ii) *If G is unimodular and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then $\|f * g\|_{L^\infty(G)} \leq \|f\|_{L^p(G)} \|g\|_{L^q(G)}$.*

Lemma 2.2 ([3])**.** *The map $T_H |_{C_c(G:H)}$ is a homeomorphism.*

The inequality stated in Lemma 2.1 (i) is said to be the Young's inequality for locally compact group G. In the following theorem we check this inequality for homogeneous spaces.

Theorem 2.3. *Let H be a compact subgroup of a locally group G. Also let p, q, r be real numbers such that $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. Then for all $\varphi \in L^p(G/H)$ and $\psi \in L^q(G/H)$ we have*

$$\|\varphi * \psi\|_{L^r(G/H)} \leq \|\varphi\|_{L^p(G/H)} \|\psi\|_{L^q(G/H)}. \tag{2.2}$$

Proof. Let φ and ψ be in $L^p(G/H)$. Then we can find two sequences $\{\varphi_n\}$ and $\{\psi_n\}$ in $C_c(G/H)$ which tend in $L^p(G/H)$-norm to φ and ψ respectively. By Lemma 2.2, there exists the sequence $\{\varphi_{n,\pi}\}$ (resp. $\{\psi_{n,\pi}\}$) in $C_c(G : H)$ such that $T_H(\varphi_{n,\pi}) = \varphi_n$ (resp. $T_H(\psi_{n,\pi}) = \psi_n$) for each natural n. Using the property of the elements of $C_c(G : H)$ and the compactness of H we have

$$\begin{aligned} T_H(|\varphi_{n,\pi}|^p)(xH) &= \int_H |\varphi_{n,\pi}(xh)|^p dh \\ &= |\varphi_{n,\pi}(x)|^p \lambda(H) = |\varphi_{n,\pi}(x)|^p. \end{aligned}$$

for all $xH \in G/H$ and $p \geq 1$ and $n \in \mathbb{N}$. Thus

$$|\varphi_n(xH)|^p = |T_H(\varphi_{n,\pi})(xH)|^p = T_H(|\varphi_{n,\pi}|^p)(xH) = |\varphi_{n,\pi}(x)|^p. \tag{2.3}$$

Now by (2.1) and (2.3) we obtain

$$\begin{aligned}
\|\varphi_n\|^p_{L^p(G/H)} &= \|T_H(\varphi_{n,\pi})\|_{L^p(G/H)} \\
&= \int_{G/H} |T_H(\varphi_{n,\pi})(xH)|^p d\mu(xH) \\
&= \int_{G/H} T_H(|\varphi_{n,\pi}|^p)(xH) d\mu(xH) \\
&= \int_G |\varphi_{n,\pi}(x)|^p d\lambda(x) = \|\varphi_{n,\pi}\|^p_{L^p(G)}.
\end{aligned} \tag{2.4}$$

Similarly for $\{\psi_n\}$ we have

$$\|\psi_n\|_{L^p(G/H)} = \|\psi_{n,\pi}\|_{L^p(G)}.$$

On the other hand $\{T_H(\varphi_{n,\pi})\}$ (resp. $\{T_H(\psi_{n,\pi})\}$) is a convergent sequence in $C_c(G/H)$ which tends to φ (resp. ψ). Now Lemma 2.2 leads us to conclude that

$$\varphi_{n,\pi} \xrightarrow{\| \|_{L^p(G)}} T_H^{-1}(\varphi),$$

$$\psi_{n,\pi} \xrightarrow{\| \|_{L^p(G)}} T_H^{-1}(\psi).$$

We denote $T_H^{-1}(\varphi)$ and $T_H^{-1}(\psi)$ by φ_π and ψ_π respectively. Now by (2.4) we have

$$\|\varphi\|_{L^p(G/H)} = \lim_{n\to\infty} \|\varphi_n\|_{L^p(G/H)} = \lim_{n\to\infty} \|\varphi_{n,\pi}\|_{L^p(G)} = \|\varphi_\pi\|_{L^p(G)}. \tag{2.5}$$

Similarly

$$\|\psi\|_{L^p(G/H)} = \|\psi_\pi\|_{L^p(G)} \tag{2.6}$$

Now Lemma 2.1, (2.5) and (2.6) give us the following result which completes the proof.

$$\begin{aligned}
\|\varphi * \psi\|_{L^r(G/H)} &= \|T_H(\varphi_\pi * \psi_\pi)\|_{L^r(G/H)} \\
&\leq \|\varphi_\pi * \psi_\pi\|_{L^r(G)} \\
&\leq \|\varphi_\pi\|_{L^p(G)} \|\psi_\pi\|_{L^q(G)} = \|\varphi\|_{L^p(G/H)} \|\psi\|_{L^q(G/H)}.
\end{aligned}$$

□

Proposition 2.4. *Let H be a compact subgroup of a unimodular locally compact group G. Let p, q be real numbers such that $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $\varphi \in L^p(G/H)$ and $\psi \in L^q(G/H)$ we have*

$$\|\varphi * \psi\|_{L^\infty(G/H)} \leq \|\varphi\|_{L^p(G/H)} \|\psi\|_{L^q(G/H)}.$$

Proof. By lemma 2.1 and (2.3) we have

$$\begin{aligned}
\|\varphi * \psi\|_{L^\infty(G/H)} &= \|T_H(\varphi_\pi * \psi_\pi)\|_{L^\infty(G/H)} \\
&= \underset{xH \in G/H}{sup} \big| T_H(\varphi_\pi * \psi_\pi)(xH) \big| \\
&= \underset{x \in G}{sup} \big| \varphi_\pi * \psi_\pi(x) \big| \\
&= \|\varphi_\pi * \psi_\pi\|_{L^\infty(G)} \\
&\leq \|\varphi_\pi\|_{L^p(G)} \|\psi_\pi\|_{L^q(G)} = \|\varphi\|_{L^p(G/H)} \|\psi\|_{L^q(G/H)}.
\end{aligned}$$

□

In [15], it has been shown that if G is compact, then $L^p(G)$ for $p \geq 1$ is a Banach algebra under the convolution (1.1). Now by Lemma 2.1 and (2.5), for all $\varphi, \psi \in L^p(G/H)$ we have

$$\begin{aligned}\|\varphi * \psi\|_{L^p(G/H)} &\leq \|T_H(\varphi_\pi * \psi_\pi)\|_{L^p(G/H)}\\ &\leq \|\varphi_\pi * \psi_\pi\|_{L^p(G)}\\ &\leq \|\varphi_\pi\|_{L^p(G)}\|\psi_\pi\|_{L^p(G)} = \|\varphi\|_{L^p(G/H)}\|\psi\|_{L^p(G/H)}.\end{aligned}$$

In particular, $L^p(G/H)$ is a Banach algebra under convolution (1.3).

Theorem 2.5. *Let $\psi \in L^p(G/H)$ for some $1 \leq p \leq \infty$ and $\varphi \in L^1(G/H)$.*

(i) *If G is unimodular, then we have*

$$\|\psi * \varphi\|_{L^p(G/H)} \leq \|\psi\|_{L^p(G/H)}\|\varphi\|_{L^1(G/H)}.$$

(ii) *If G is not unimodular, we still have $\psi * \varphi \in L^p(G/H)$ if $\psi_\pi \in C_c(G/H)$.*

Proof. (i) Using Theorem 2.39 of [2] and (2.5) we conclude

$$\begin{aligned}\|\psi * \varphi\|_{L^p(G/H)} &\leq \|T_H(\psi_\pi * \varphi_\pi)\|_{L^p(G/H)}\\ &\leq \|\psi_\pi * \varphi_\pi\|_{L^p(G)}\\ &\leq \|\psi_\pi\|_{L^p(G)}\|\varphi_\pi\|_{L^1(G)} = \|\psi\|_{L^p(G/H)}\|\varphi\|_{L^1(G/H)}.\end{aligned}$$

(ii) The proof of (ii) is similar to (i). □

The L^p-conjecture states that if $f * g$ exists and belongs to $L^p(G)$ for all $f, g \in L^p(G)$, then G is compact. This conjecture was first expressed specifically by Rajagopalan in his Ph.D thesis in 1963. But before him, Zelazko [14] and Urbanik [13] had proved the first result related to this conjecture in 1961; they showed that the conjecture is true for all locally compact abelian groups. The truth of the conjecture is proved for $p > 2$ by Zelazko [15] and Rajagopalan [8] independently. Also Rajagopalan has established it for the case that $p \geq 2$ and G is discrete [7], for the case that $p = 2$ and G is totally disconnected [8] and for the case that $p > 1$ and G is either nilpotent or a semidirect product of two locally compact groups [9]. In a joint work [10] they confirmed the conjecture for $p > 1$ and amenable groups. Rikert [11] showed the conjecture is true for the case $p = 2$. Finally, Saeki [12] gave an affirmative answer to the conjecture in 1990. As a consequence of our results, we can state the L^p-conjecture on homogeneous spaces.

Corollary 2.6. *Let H be a compact subgroup of locally compact group G. Then $L^p(G/H)$ is closed under the convolution defined in* (1.1) *if and only if G/H is compact.*

References

[1] A. Arefijamaal, *The continuous Zak transform and generalized Gabor frames*, Mediterr. J. Math., **10** (2013), 353–365. 2

[2] G. B. Folland, *A Course in Abstract Harmonic Analysis*, CRC Press, Boca Raton, (1995). 1, 2.1, 2

[3] A. Ghaani Farashahi, *Convolution and involution on function spaces of homogeneous spaces*, Bull. Malays. Math. Sci. Soc., **36** (2013), 1109–1122. 1, 1, 2.2

[4] E. Hewitt, K. A. Ross, *Abstract harmonic analysis*, Springer-Verlag, Berlin, (1963). 2.1

[5] R. A. Kamyabi-Gol, N. Tavallaei, *Convolution and homogeneous spaces*, Bull. Iranian Math. Soc., **35** (2009), 129–146. 1

[6] R. A. Kamyabi-Gol, N. Tavallaei, *Wavelet transforms via generalized quasi-regular representations*, Appl. Comput. Harmon. Anal., **26** (2009), 291–300. 1

[7] M. Rajagopalan, *On the L^p-spaces of a discrete group*, Colloq. Math., **10** (1963), 49–52. 2

[8] M. Rajagopalan, *L^p-conjecture for locally compact groups I*, Trans. Amer. Math. Soc., **125** (1966), 216–222. 2

[9] M. Rajagopalan, *L^p-conjecture for locally compact groups II*, Math. Ann., **169** (1967), 331–339. 2

[10] M. Rajagopalan, W. Zelazko, *L^p-conjecture for solvable locally compact groups*, J. Indian Math. Soc., **29** (1965) 87–93. 2

[11] N. W. Rickert, *Convolution of L^2 functions*, Colloq. Math., **19** (1968), 301–303. 2

[12] S. Saeki, *The L^p-conjecture and Young's inequality*, Illinois J. Math., **34** (1990), 615–627. 2

[13] K. Urbanik, *A proof of a theorem of Zelazko on L^p-algebras*, Colloq. Math., **8** (1961), 121–123. 2

[14] W. Zelazko, *On the algebras L^p of locally compact groups*, Colloq. Math., **8** (1961), 112–120. 2

[15] W. Zelazko, *A note on L^p-algebras*, Colloq. Math., **10** (1963), 53–56. 2, 2