Available online at www.vonneumann-publishing.com/fatma
Func. Anal.-TMA 1 (2015), 50–55
Research Article

Functional Analysis: Theory, Methods & Applications
Journal Homepage: http://vonneumann-publishing.com/fatma

# The mean ergodic theorem for nonexpansive mappings in p-Banach spaces

Hassan M. Kenari[a,*], Yeol Je Cho[b], Themistocles M. Rassias[c], Reza Saadati[d]

[a]*Department of Mathematics, Science and Research Branch, Islamic Azad University, Post Code 14778, Ashrafi Esfahani Ave, Tehran, I. R. Iran.*

[b]*Department of Mathematics Education and the RINS, Gyeongsang National University, Jinju 660-701, Korea, and Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia.*

[c]*Department of Mathematics, National Technical University of Athens, Zografou Campus, 15780 Athens, Greece.*

[d]*Department of Mathematics and Computer Science, Iran University of Science and Technology, Tehran, Iran.*

Communicated by Y. Yang

**Abstract**

In this paper, we present a mean ergodic theorem for nonexpansive mappings in p-Banach spaces. Our results extended and generalized some results of [8].

*Keywords:* Ergodic theorem, nonexpansive mappings, p-Banach space.
*2010 MSC:* Primary 39A10, 39B72; Secondary 47H10, 46B03.

## 1. Introduction

Let $X$ be a Banach space and $C$ be a closed convex subset of $X$. The mapping $T : C \to C$ is called nonexpansive on $C$ if

$$\|Tx - Ty\| \le \|x - y\|, \qquad \forall x, y \in C.$$

Let $F(T)$ be the set of fixed points of $T$. If $X$ is strictly convex, $F(T)$ is closed and convex. In [1], Baillon proved the first nonlinear ergodic theorem such that if $X$ is a real Hilbert space and $F(T) \neq \emptyset$, then for each $x \in C$, the sequence $\{S_n x\}$ defined by

$$S_n x = \left(\frac{1}{n}\right)(x + Tx + \cdots + T^{n-1}x),$$

*Corresponding author

*Email addresses:* hmkenari@@hush.ai (Hassan M. Kenari), yjcho@@gnu.ac.kr (Yeol Je Cho), trassias@@math.ntua.gr (Themistocles M. Rassias), rsaadati@@eml.cc (Reza Saadati)

*Received 2015-11-29*

converges weakly to a fixed point of $T$. It was also shown by Pazy [11] that if $X$ is a real Hilbert space and $S_n x$ converges weakly to $y \in C$, then $y \in F(T)$. These results were extended by Baillon [2], Bruck [5] and Reich [13], [14] and [15]. Our results generalized the recent results of [8, 9].

## 2. p-norm

**Definition 2.1** (see [3, 16]). Let $X$ be a real linear space. A function $\|.\| : X \to \mathbb{R}$ is a quasi-norm (valuation) if it satisfies the following conditions:
(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;
(2) $\|\lambda x\| = |\lambda|.\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$;
(3) There is a constant $M \geq 1$ such that $\|x + y\| \leq M(\|x\| + \|y\|)$ for all $x, y \in X$.
Then $(X, \|.\|)$ is called a quasi-normd space. The smallest possible $M$ is called the modulus of concavity of $\|.\|$. A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $\|.\|$ is called a p-norm $0 < p < 1$ if

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p,$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p-Banach space.
By the Aoki-Rolewicz [16], each quasi-norm is equivalent to some p-norm (see also [8] and [10, 12, 17, 18, 19, 20]).
Since it is much easier to work with p-norm, we restrict our attention mainly to p-norms.

## 3. Main results

To prove the main results in this paper, first, we introduce some lemmas.

**Definition 3.1.** The modulus of convexity of a Banach space $(X, \|.\|)$ is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \|\frac{x + y}{2}\| : x, y \in S, \|x - y\| \geq \epsilon \right\},$$

where $S$ denotes the unit sphere of $(X, \|.\|)$ in the definition of $\delta(\epsilon)$, one can as well take the infimom over all vectors $x, y \in X$ such that $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$. The characteristic of convexity of the space $(X, \|.\|)$ is the number $\epsilon_0$ defined by

$$\epsilon_0 = \sup\{\epsilon : \delta(\epsilon) = 0\}.$$

**Lemma 3.2** ([7]). *Let $(X, \|.\|_p)$, be a uniformly convex Banach space with modulus of convexity $\delta$. Let $x, y \in X$. If $\|x\|_p \leq r$, $\|y\|_p \leq r$, $r \leq R$ and $\|x - y\|_p \geq \epsilon > 0$, then*

$$\|\lambda x + (1 - \lambda)y\|_p \leq r(1 + |2\lambda|^p(1 - \delta_R(\epsilon)),$$

*for all $\lambda : 0 \leq \lambda \leq 1$, where $\delta_R(\epsilon) = \delta(\frac{\epsilon}{R})$.*

*Proof.* Since $\lambda \geq \frac{1}{2}$, implies $(1 - \lambda) \leq \frac{1}{2}$, we may assume that $\lambda \leq \frac{1}{2}$. We then have

$$\begin{aligned} \|\lambda x + (1 - \lambda)y\|_p = \left\| 2\lambda \frac{(x + y)}{2} + (1 - 2\lambda)y \right\|_p &\leq r(|2\lambda|^p(1 - \delta_R(\epsilon)) + |1 - 2\lambda|^p) \\ &\leq r(1 + |2\lambda|^p(1 - \delta_R(\epsilon))). \end{aligned}$$

□

**Lemma 3.3.** *Let $C$ be a closed convex subset of $X$ and $T : C \to C$ be nonexpansive mappings. Let $x \in C$, $f \in F(T)$ and $0 < \alpha \leq \beta < 1$. Then, for any $\epsilon > 0$, there exists $N > 0$ such that for all $n \geq N$,*

$$\|T^k(\lambda T^n x + (1-\lambda)f) - (\lambda T^{n+k}x + (1-\lambda)f))\|_p < \epsilon,$$

*for all $k > 0$ and $\lambda : \alpha \leq \lambda \leq \beta$.*

*Proof.* Put $r = \lim_n \|T^n x - f\|_p$, $R = \|x - f\|_p$, and $c = \min\{|2\lambda|^p : \alpha \leq \lambda \leq \beta\}$. For given $\epsilon > 0$, choose $d > 0$ such that $\frac{r}{r+d} > 1 + c(1-\delta_R(\epsilon))$. Then there exists $N > 0$ such that, for all $n \geq N$, $\|T^n x - f\|_p < r + d$.

For each $n \geq N$ , $k > 0$ and $\alpha \leq \lambda \leq \beta$, we put $u = (1-\lambda)(T^k z - f)$ and $v = \lambda(T^{n+k}x - T^k z)$ where $z = \lambda T^n x + (1-\lambda)f$. Then we have $\|u\|_p \leq |\lambda|^p |(1-\lambda)|^p \|T^n x - f\|_p$ and $\|v\|_p \leq |\lambda|^p |(1-\lambda)|^p \|T^n x - f\|_p$. Suppose that $\|u - v\|_p = \|T^k z - (\lambda T^{n+k}x + (1-\lambda)f)\|_p \geq \epsilon$. So by Lemma 3.2, we have

$$\begin{aligned}\|\lambda u + (1-\lambda)v\|_p &= |\lambda|^p |(1-\lambda)|^p \|T^{n+k}x - f\|_p \\ &\leq |\lambda|^p |(1-\lambda)|^p \|T^n x - f\|_p (1 + |2\lambda|^p (1 - \delta_R(\epsilon)) \\ &\leq |\lambda|^p |(1-\lambda)|^p \|T^n x - f\|_p (1 + c(1 - \delta_R(\epsilon))).\end{aligned}$$

Hence we have $(r + d)(1 + c(1 - \delta_R(\epsilon))) < r \leq (r + d)(1 + c(1 - \delta_R(\epsilon)))$, which is a contradiction. This completes the proof. □

**Lemma 3.4** (Browder [4])**.** *Let $C$ be a closed convex subset of $X$ and $T : C \to C$ be a nonexpansive mapping. If $\{u_i\}$ is a weakly convergent sequence in $C$ with weak limit $u_0$ and if $\lim_i \|u_i - Tu_i\|_p = 0$, then $u_0$ is a fixed point of $T$.*

**Lemma 3.5.** *Let $C$ be a closed convex subset of $X$ and $T : C \to C$ be a nonexpansive mapping. Then for all $x \in C$ and $n > 0$,*

$$\lim_{i\to\infty} \|T^k S_n T^i x - S_n T^k T^i x\|_p = 0, \tag{3.1}$$

*uniformly for each $k \geq 1$.*

*Proof.* By induction on $n$, we prove this lemma. First, we prove the conclusion in the case $n = 2$. Put $r = \lim\limits_{n\to\infty} \|T^{n+1}x - T^n x\|_p$, $R = \|x - Tx\|_p$ and $x_i = T^i x$ for $i \geq 1$.

If $r \neq 0$, then, for any $\epsilon > 0$ , choose $c > 0$ such that $\frac{r}{r+c} > 1 - \delta_R(\epsilon)/2$. Then there exists $N > 0$ such that, for all $i \geq N$, $\|T^k x_i - T^{k+1} x_i\|_p \leq r + c$ for each $k \geq 1$. If we put $u = \frac{1}{2}(T^k z - T^k x_i)$ and $v = \frac{1}{2}(T^{k+1}x_i - T^k z)$ where $i \geq N$, $k > 0$ and $z = \frac{1}{2}(x_i + Tx_i)$, then we have

$$\|u\|_p \leq \left(\frac{1}{2}\right)^p \|z - x_i\|_p = \left(\frac{1}{4}\right)^p \|Tx_i - x_i\|_p \leq \left(\frac{1}{4}\right)^p (r + c).$$

Similarly, we have $\|v\|_p \leq (\frac{1}{4})^p (r + c)$. Suppose that $\|u - v\|_p = \|T^k z - \frac{1}{2}(T^{k+1}x_i + T^k x_i)\|_p \geq \epsilon$. Then, we have

$$\left\|\frac{1}{2}(u + v)\right\|_p = \left(\frac{1}{4}\right)^p \|T^{k+1}x_i - T^k x_i\|_p \leq \left(\frac{1}{4}\right)^p (r + c)\left(1 - \frac{1}{2}\delta_R(\epsilon)\right),$$

which contradicts $r > (r + c)(1 - \frac{1}{2}\delta_R(\epsilon))$.

If $r = 0$, then for any $\epsilon > 0$, choose $i > 0$ so large that $\|u\|_p < \frac{\epsilon}{2}$ and $\|v\|_p < \frac{\epsilon}{2}$ Hence we have $\|T^k z - \frac{1}{2}(T^{k+1}x_i + T^k x_i)\|_p = \|u - v\|_p \leq \|u\|_p + \|v\|_p < \epsilon$. This completes the proof of the case $n = 2$. Now, suppose that $\lim\limits_{i\to\infty} \|T^k S_{n-1} x_i - S_{n-1} T^k x_i\|_p = 0$, uniformly for each $k \geq 1$. We clime that $\lim\limits_{i\to\infty} \|S_{n-1} T x_i - x_i\|_p$ exist. Put $r = \liminf\limits_{i\to\infty} \|S_{n-1} T x_i - x_i\|_p$. For any $\epsilon > 0$, choose $i > 0$ such that $\|S_{n-1} T x_i - x_i\|_p < r + \frac{\epsilon}{2}$ and $\|S_{n-1} T^k x_{i+1} - T^k S_{n-1} x_{i+1}\|_p < \frac{\epsilon}{2}$. Then we have

$$\begin{aligned}\|S_{n-1} T x_{i+k} - x_{i+k}\|_p &\leq \|S_{n-1} T^k x_{i+1} - T^k S_{n-1} x_{i+1}\|_p + \|T^k S_{n-1} x_{i+1} - T^k x_i\|_p \\ &< \frac{\epsilon}{2} + r + \frac{\epsilon}{2} = r + \epsilon,\end{aligned}$$

for all $k \geq 1$. Then, we have

$$\limsup_{i\to\infty} \|S_{n-1}Tx_i - x_i\|_p = \limsup_{k\to\infty} \|S_{n-1}Tx_{i+k} - x_{i+k}\|_p < r + \epsilon.$$

Since $\epsilon$ is arbitrary, we have

$$\limsup_{i\to\infty} \|S_{n-1}Tx_i - x_i\|_p \leq \liminf_{i\to\infty} \|S_{n-1}Tx_i - x_i\|_p,$$

i.e., $\lim_{i\to\infty} \|S_{n-1}Tx_i - x_i\|_p$ exists. Now we put $r = \lim_{i\to\infty} \|S_{n-1}Tx_i - x_i\|_p$. If $r \neq 0$, Then, for any $\epsilon$, choose $c > 0$ such that $\frac{(r-c)}{(r+2c)} > 1 - (2\frac{(n-1)}{n^2})\delta_{3r}(\epsilon)$. Then there exists $N > 0$ such that, if, for all $i \geq N$, $|\|S_{n-1}Tx_i - x_i\| - r| \leq c$ and $\|S_{n-1}T^k x_{i+1} - T^k S_{n-1}x_{i+1}\|_p \leq \frac{c}{n^p}$ , we put $u = \left(\frac{n}{(n-1)}\right)(T^k S_n x_i - T^k x_i)$ and $v = n(S_{n-1}T^k x_{i+1} - T^k S_n x_i)$. so

$$\|u\|_p \leq \|S_{n-1}Tx_i - x_i\|_p \leq r + c,$$

$$\|v\|_p \leq (n)^p \|S_{n-1}T^k x_{i+1} - T^k S_{n-1}x_{i+1}\|_p + \|S_{n-1}Tx_i - x_i\|_p \leq r + 2c,$$

and

$$\|u - v\|_p = \left(\frac{n}{n-1}\right)^p \|T^k S_n x_i - S_n T^k x_i\|_p.$$

Hence, by the method in the proof of the case $n = 2$, we have $\|T^k S_n x_i - S_n T^k x_i\|_p < \epsilon$ for all $k \geq 1$, and $i \geq N$. If $r = 0$, then, as in the proof of the case $n = 2$, there exists $N'$ such that, for each $i \geq N'$, $\|u\|_p < \frac{\epsilon}{2}$ , $\|v\|_p < \frac{\epsilon}{2}$. Therefore, we have $\|T^k S_n x_i - S_n T^k x_i\|_p < \epsilon$. This completes the proof. □

Now, assume that the norm of $X$ is Frechet differentiable.

**Definition 3.6.** Let $X$ be a normed space and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of elements in $X$. Recall that we say that the sequence converges to $x \in X$,

$$x_n \to x \quad as \quad n \to \infty \quad if \quad \|x_n - x\| \to 0 \quad as \quad n \to \infty.$$

We say that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to $x \in X$, and write

$$x_n \to^w x$$

if for all $\phi \in X^*$, we have

$$\phi(x_n) \to \phi(x) \quad as \quad n \to \infty.$$

**Proposition 3.7** (cf. [5], [14], [8])**.** *Let $C$ be a closed convex subset of $X$ and $T : C \to C$ be a nonexpansive mapping. If we put $W(x) = \cap_m \overline{co}\{T^k x : k \geq m\}$ for all $x \in C$, then $W(x) \cap F(T)$ is at most one point.*

*Proof.* Suppose that $f, g \in W(x) \cap F(T)$ and $f \neq g$. Put $h = \frac{f+g}{2}$ and $r = \lim_{n\to\infty} \|T^n x - g\|_p$. since $h \in W(x)$, $\|h - g\|_p \leq r$. For each $n$, we choose $p_n \in [T^n x, h]$ such that

$$\|p_n - g\|_p = \min\{\|y - g\|_p : y \in [T^n x, h]\}.$$

By Theorem 2.5 of [6], $(J(g - p_n), p_n - T^n x) \geq 0$ where $J$ is the duality mapping. Since $p_n \in [T^n x, h]$, we have $(J(g - p_n), h - T^n x) \geq 0$. Suppose that

$$\liminf_{n\to\infty} \|p_n - g\|_p = \|h - g\|_p.$$

Since $X$ is uniformly convex and $\|p_n - g\|_p \leq \|\frac{(p_n+h)}{2} - g\|_p \leq \|h-g\|_p$,$p_n$ converges strongly to $h$. Since the duality mapping $J$ is norm-to-norm continuous, we have that for given $\epsilon > 0$, there exists $N > 0$ such that $(J(g-h) - J(g-p_n), h - T^n x) \geq -\epsilon$, for all $n \geq N$. Therefore we have

$$\begin{aligned}(J(g-h), h-T^n x) &= (J(g-h) - J(g-p_n), h-T^n x) + (J(g-p_n), h-T^n x)\\ &\geq -\epsilon + 0 = -\epsilon.\end{aligned}$$

Then it follows that for each $y \in \cap_m \overline{co}\{T^k x : k \geq m\}$, $(J(g-h), h-y) \geq 0$.If we put $y = g$, we have $\|h-g\|_p = 0$. This contradicts $h \neq g$. Suppose that $\liminf_{n\to\infty} \|p_n - g\|_p < \|h-g\|_p$, then there exist $c > 0$ and a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ such that $\|p_{n_i} - g\|_p + c < \|h-g\|_p$. Put $p_{n_i} = \alpha_i T^{n_i} x + (1-\alpha_i)h$, for $i = 1, 2, \ldots$. Then there exist $\alpha > 0$ and $\beta < 1$ such that $\alpha \leq \alpha_i \leq \beta$ for all $i$. By Lemma 3.3, there exist $N > 0$ such that if $n \geq N$,

$$\|T^k(\lambda T^n x + (1-\lambda)h) - (\lambda T^{n+k} x + (1-\lambda)h\|_p < c,$$

for all $\lambda : \alpha \leq \beta$ and for all $k > 0$. If we choose $p_{n_{i_0}} \in \{p_{n_i}\}$ such that $n_{i_0} \geq N$, we have

$$\begin{aligned}\|p_{n_{i_0}+k} - g\|_p &= \|(\alpha_{i_0} T^{n_{i_0}+k} x + (1-\alpha_{i_0})h) - g\|_p\\ &\leq \|T^k p_{n_{i_0}} - (\alpha_{i_0} T^{n_{i_0}+k} x + (1-\alpha_{i_0})h)\|_p + \|T^k p_{n_{i_0}} - g\|_p\\ &< c + \|p_{n_{i_0}} - g\|_p < \|h-g\|_p,\end{aligned}$$

for $k = 1, 2, \ldots$. Therefore we have $p_n \neq h$ for all $n \geq n_{i_0}$. It follows that $(J(g-h), h-T^n x) \leq 0$ for $n \geq n_{i_0}$. Then we have $J(g-h), h-y) \leq 0$ for all $y \in \overline{co}\{T^k x : k \geq n_{i_0}\}$. Put $y = f = h + (h-g)$, then $\|h-g\|_p = 0$. This contradicts $h \neq g$. □

In this paper, we give a new proof of the following theorem, which is due to Reich [14].

**Theorem 3.8.** *Let $(X, \|.\|_p)$ be a uniformly convex Banach space which has the Frechet differentiable norm. Let $C$ be a closed convex subset of $X$ and, $T : C \to C$ be a nonexpansive mapping. Then the following statements are equivalent:*
*(1) $F(T) \neq \emptyset$;*
*(2) $\{T^n x\}$ is bounded for each $x \in C$;*
*(3) for all $x \in C$, $S_n T^i x$ converges weakly to a point $y \in C$, uniformly for each $i \geq 1$.*

*Proof.* (1) $\Leftrightarrow$ (2) is well known in [4].
(2) $\Leftrightarrow$ (3) Suppose that, for some $x \in C$, there exist an unbounded subsequence $\{T^{n_i} x\}$ of $\{T^n x\}$. Since $T$ is nonexpansive mapping, it follows that, for each $m > 0$, the sequence $\{S_m T^{n_i} x\}$ is also unbounded, which contradicts the condition (3).
(2) $\Leftrightarrow$ (3) Since $\{T^n x\}$ is bounded and

$$\begin{aligned}\|TS_n T^i x - S_n T^i x\|_p &\leq \|TS_n T^i x - S_n TT^i x\|_p + \|S_n TT^i x - S_n T^i x\|_p\\ &\leq \|TS_n T^i x - S_n TT^i x\|_p + \left(\frac{1}{n}\right)^p \|T^{i+1+n} x - T^i x\|_p,\end{aligned}$$

there exists a sequence $\{S_n T^{i_n} x\}$ such that

$$\lim_{n\to\infty} \|TS_n T^{i_n} x - S_n T^{i_n} x\|_p = 0.$$

Then by Lemma 3.3, it follows that any weakly p-convergent subsequence of $\{S_n T^{i_n} x\}$ p-converges weakly to a point $y$, i.e., $S_n T^{i_n} x \rightharpoonup y$, where $y = W(x) \cap F(T)$. Also, by Lemma 3.4, it follows that

$$\lim_{n\to\infty} \|TS_n T^{i_n+kn+i} x - S_n T^{i_n+kn+i} x\|_p = 0,$$

for all $i, k \geq 1$. Therefore, $S_n T^{i_n+kn} x_i \to y$ uniformly for each $k \geq 1$.
On the other hand, for each $n \geq 1$ with $m \geq i_n$, we have

$$S_m T^i x = \frac{1}{m} \sum_{k=0}^{m-1} T^k x_i = \frac{1}{m} \left( \sum_{k=i_n+tn}^{m-1} T^k x_i + n \left( \sum_{k=0}^{t} S_n T^{i_n+kn} x_i \right) + \sum_{k=0}^{i_n} T^k x_i \right),$$

where $m = tn + i_n + r$, $r < n$. Since $\{S_n T^{i_n+kn} x_i\}$ p-converges to $y$ uniformly for each $k \geq 1$, it follows that $S_m T^i x$ converges weakly to $y$, uniformly for each $i \geq 1$. This completes the proof.

□

## References


[1] J. B. Baillon, *Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris Ser. A-B, **280** (1975), 1511–1514. 1

[2] J. B. Baillon, *Comportement asymptotique des iteres de contractions non lineaires dans les espace $L_p$*, C. R. Acad. Sci. Paris ser. A-B, **286** (1978), 157–159. 1

[3] Y. Benyamini, J. Lindenstrauss, *Geometric nonlinear functional analysis*, Amer. Math. Soc., Colloq. Publ., (2000). 2.1

[4] F. E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach space*, Proc. Amer. Math. Soc., (1976). 3.4, 3

[5] R. E. Bruk, *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach space*, Isr. J. Math., **32** (1979), 107–116. 1, 3.7

[6] F. R. Deutsch, P. H. Maserick, *Applications of the Hahn-Banach theorem in approximation theory*, SIAM Rev., **9** (1967), 516–530. 3

[7] C. W. Groetsch, *A note on segmenting Mann iterates*, J. Math. Anal. Appl., **40** (1972), 369–372. 3.2

[8] N. Hirano, *A proof of the mean ergodic theorem for nonexpansive mappings in Banach space*, Proc. Amer. Math. Soc., **78** (1980), 361–365. (document), 1, 2, 3.7

[9] H. M. Kenari, R. Saadati, Y. J. Cho, *The mean ergodic theorem for nonexpansive mappings in multi-Banach spaces*, J. Inequal. Appl., **2014** (2014), 9 pages. 1

[10] P. Li, S. M. Kang, L. J. Zhu, *Visco-resolvent algorithms for monotone operators and nonexpansive mappings*, J. Nonlinear Sci. Appl., **7** (2014), 325–344. 2

[11] A. Pazy, *on the asymptotic behavior of iterates of nonexpansive mappings in Hilbert space*, Isr. J. Math., **26** (1977), 197–204. 1

[12] Y. Purtas, H. Kiziltunc, *Weak and strong convergence of an explicit iteration process for an asymptotically quasi-i-nonexpansive mapping in Banach spaces*, J. Nonlinear Sci. Appl., **5** (2012), 403–411. 2

[13] S. Reich, *Almost convergence and nonlinear ergodic theorems*, J. Approx. Theory, **24** (1978), 296–272. 1

[14] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach space*, J. Math. Anal. Appl., **67** (1979), 274–276. 1, 3.7, 3

[15] S. Reich, *Nonlinear ergodic theory in Banach space*, Argonne National Laboratory Report, (1979), 69–79. 1

[16] S. Rolewicz, *Metric linear space*, D. Reidel, (1985). 2.1, 2

[17] G. S. Saluja, *Weak convergence theorems for two asymptotically quasi-nonexpansive non-self mappings in uniformly convex Banach spaces*, J. Nonlinear Sci. Appl., **7** (2014), 138–149. 2

[18] T. Thianwan, *Convergence criteria of modified Noor iterations with errors for three asymptotically nonexpansive nonself-mappings*, J. Nonlinear Sci. Appl., **6** (2013), 181–197. 2

[19] E. Yolacan, H. Kiziltunc, *On convergence theorems for total asymptotically nonexpansive nonself-mappings in Banach spaces*, J. Nonlinear Sci. Appl., **5** (2012), 389–402. 2

[20] K. Zennir, *Exponential growth of solutions with $L_p$-norm of a nonlinear viscoelastic hyperbolic equation*, J. Nonlinear Sci. Appl., **6** (2013), 252–262. 2