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Existence of solution for fractional impulsive integro-differential equation with integral boundary conditions

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Abstract

In this paper we have established the existence and uniqueness of solution for a class of impulsive fractional integro-differential equation with nonlocal jump type integral boundary conditions. The results of the paper are obtained by applying the Banach and Krasnoselkii's fixed point theorems. At last an application is given to verify our results.

Keywords: Fractional derivative, impulsive, integral boundary conditions, jump conditions.
2010 MSC: 34A37, 34A08, 26A33, 34B10.

1. Introduction

Recently it has been shown that some physical and biological systems can be modeled more accurately using some fractional derivatives. The non integer type models have emerged as a popular field of research due to its extensive development and applications in several disciplines such as physics, mechanics, chemistry, engineering, etc. For more details we refer the book of A. A. Kilbas *et al.* [10], and paper of R. P. Agarwal *et al.* [1] and references therein.

Integral type nonlocal boundary conditions can be seen in models of a variety of science and engineering disciplines. For examples: heat conduction, chemical engineering, thermo-elasticity, and plasma physics. For details we cite the papers [1, 2, 3, 5, 6, 15, 16, 22, 23]. Impulsive fractional differential equations play an important role in realistic description of observed evolution phenomena of several physical problems. For examples: Fluctuations of pendulum system in the case of external impulsive effects, percussive model of a clock mechanism, and dynamic of a system with automatic regulation e.t.c. For more details we refer the papers [3, 4, 5, 6, 7, 9, 11, 12, 21].

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The jump conditions are very universal and include many conditions as special cases. Such type of conditions are mostly seen in engineering and many of the physical systems such as population dynamics, blood flow models, chemical engineering, biology and cellular systems etc. Due to their significance many author's have been established the results for solvability of the problems with such type of conditions. For more details we refers the reader to [14, 17].

In [20] Y. Wang *et al.* study integral boundary value problem involving Caputo differentiation of order $q \in (1, 2)$. By using some fixed point theorems we prove the existence of positive solutions.

$$\begin{cases} D^q u(t) = f(t, u(t)),\ t \in (0, 1), \\ \alpha u(0) - \beta u(1) = \int_0^1 h(t)u(t)dt,\ \gamma u'(0) - \delta u'(1) = \int_0^1 g(t)u(t)dt, \end{cases} \tag{1.1}$$

where $\alpha,\ \beta,\ \gamma,\ \delta$, are constants with $\alpha > \beta > 0,\ \gamma > \delta > 0$.

In [8] Xi Fu *et al.* concerned the following fractional separated boundary value problem with fractional impulsive conditions:

$$\begin{cases} {}^cD^\alpha x(t) = f(t, x(t))\ t \in [0, T],\ t \neq t_k,\ k = 1, 2, \dots, m,\ \alpha \in (1, 2), \\ \Delta x(t_k) = I_k(x(t_k^-)),\ \Delta({}^cD^\gamma x(t_k)) = I_k{}^*(x(t_k^-)),\ k = 1, 2, \dots, m,\ \gamma \in (0, 1), \\ a_1 x(0) + b_1({}^cD^\gamma x(0)) = c_1,\ a_2 x(T) + b_2({}^cD^\gamma x(T)) = c_2, \end{cases} \tag{1.2}$$

where $a_i, b_i, c_i, \in R,\ i = 1, 2$, with $a_i \neq 0$ and $a_2 T^\gamma \Gamma(2 - \gamma) \neq -b_2$. By using the Schaefer, Banach, and Nonlinear alternative of Leray-Schauder theorems, author's obtained the existence results.

In [18] C. Thaiprayoon *et al.* established the existence of extremal solutions for the periodic boundary value problems for second-order impulsive integro-differential equations with integral type jump conditions. Subsequently, In [19] C. Thaiprayoon *et al.* study the following impulsive fractional boundary-value problems with fractional integral jump conditions:

$$\begin{cases} {}^cD^\alpha u(t) = f(t, u(t)),\ 0 < \alpha \leq 1,\ t \in [0, T], \\ \Delta u(t_k) = J_k\Big(\sum_{j=1}^k d_{k,j} I^{\beta_{k,j}} u(t_j^-)\Big),\ k = 1, 2, \dots, m, \\ au(0) + bu(T) = c, \end{cases} \tag{1.3}$$

by using a variety of fixed-point theorems, author's proved some new existence and uniqueness results.

Motivated by the above defined works, we investigate the existence and uniqueness of solution for the following class of fractional impulsive integro-differential equations with integral boundary conditions:

$${}^cD^\alpha u(t) = f(t, u(t)) + \int_0^t g(t - s)p(s, u(s))ds,\ t \in [0, T], \tag{1.4}$$

$$\Delta u(t_k) = S_k\Big(\sum_{j=1}^k c_{k,j} I_t^{\beta_{k,j}} u(t_j^-)\Big),\ k = 1, 2, \dots, m, \tag{1.5}$$

$$\Delta({}^cD^q u(t_k)) = J_k\Big(\sum_{j=1}^k d_{k,j} I_t^{\beta_{k,j}} u(t_j^-)\Big),\ 0 < q < 1,\ k = 1, 2, \dots, m, \tag{1.6}$$

$$u(0) + h(u) = \int_0^T q_1(u(s))ds,\ D^q u(0) + D^q u(T) = \int_0^T q_2(u(s))ds, \tag{1.7}$$

where ${}^cD^\alpha$, is the Caputo's derivative of order $\alpha \in (1, 2)$. The functions $f : [0, T] \times X \to X,\ g : X \to X,\ p : [0, T] \times X \to X$ and $q_1, q_2 : X \to X$ are given continuous functions. The impulsive conditions for $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $S_k, J_k \in C(X, X)$, are continuous and bounded functions, $c_{k,j}, d_{k,j}$ are positive constants, $I_t^{\beta_{k,j}}$ is the Riemann-Liouville fractional integral operator of order $\beta_{k,j} > 0$ for $j = 1, 2, \dots, k$ and $k = 1, 2, \dots, m$. The notations are defined as $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ and $\Delta({}^cD^q u(t_k)) = ({}^cD^q u(t_k^+)) -$

$({}^cD^q u(t_k^-))$, $u(t_k^+)$ and $u(t_k^-)$ represents the right and left-hand limits of $u(t)$ at $t = t_k$ respectively with $u(t_i^-) = u(t_i)$.

In this paper we establish the existence and uniqueness results for nonlinear fractional integro-differential equations subject to integral type boundary conditions with jump impulsive conditions by using some fixed point theorems. Our aim is to deal jump integral conditions (1.5)–(1.6), these conditions means that a sudden change of values of $u(t)$ and its derivative at impulsive points t_k depend on the area under the curves of $u(t_k)$ and ${}^cD^q u(t_k)$. It should be notice that the impulsive effect of the system (1.4)–(1.7) has memory of the past states. The rest of the work is structured as follows: In Section 2, we present some basic definitions and defined the required space. In Section 3, we discuss the existence and uniqueness results for solutions of the system (1.4)–(1.7), by using the Banach and Krasnoselkii's fixed point theorems.

2. Preliminaries

Let $(X, \|\cdot\|_X)$ be a complex Banach space of functions with the norm $\|y\|_X = \sup_{t\in[0,T]}\{|y(t)| : y \in X\}$. The space $L^p([0,T], X)$ stand for the Banach space of all Lebesgue measurable functions $\Omega : [0,T] \to X$ with $\|\Omega\|_{L^p([0,T])} < \infty$. To treat the impulsive conditions, we define the following spaces

$$PC_t = PC([0,t] : X),\ 0 \le t \le T,$$

be a Banach space of all such functions $y : [0,T] \to X$, which are continuous every where except for a finite number of points $t_i,\ i = 1,2,\dots,m$, at which $y(t_i^+)$ and $y(t_i^-)$ exists with $y(t_i^-) = y(t_i)$ and endowed with the norm

$$\|y\|_{PC_t} = \sup_{t\in[0,T]}\{\|y(t)\|_X, y \in PC_t\}.$$

Further, we define the space

$$PC_t^1 = PC^1([0,t] : X),\ 0 \le t \le T,$$

be a Banach space of all such functions $y : [0,T] \to X$, which are continuously differentiable every where except for a finite number of points $t_i,\ i = 1,2,\dots,m$, at which $y'(t_i^+)$ and $y'(t_i^-)$ exists with $y'(t_i^-) = y'(t_i)$ and endowed with the norm

$$\|y\|_{PC_t^1} = \sup_{t\in[0,T]}\{\|y(t)\|_{PC_t}, \|y'(t)\|_{PC_t}, y \in PC_t\}.$$

Rest of the notations used in the paper have their usual meanings if not specified.

Definition 2.1 ([7])**.** The fractional integral of order α with lower limit zero for a function $f : [0,\infty) \to R$ is defined as

$$I_t^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds,\ t > 0,\ \alpha > 0, \tag{2.1}$$

provided the right side is point-wise defined on $[0,\infty)$, where Γ is the gamma function.

Definition 2.2 ([7])**.** The Riemann-Liouville derivative of order α with the lower limit zero for a function $f : [0,\infty) \to R$ can be written as

$${}^LD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\Big(\frac{d}{dt}\Big)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds,\ t > 0,\ n-1 < \alpha < n. \tag{2.2}$$

Definition 2.3 ([7])**.** The Caputo's derivative of order α for a function $f : [0,\infty) \to R$ can be written as

$${}^cD_t^\alpha f(t) = {}^L D_t^\alpha\Big[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)\Big],\ t > 0,\ n-1 < \alpha < n. \tag{2.3}$$

Remark 2.4 ([7]). If $f(t) \in C^n[0,\infty)$, for order $n-1<\alpha<n$ then

$$^cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds = I_t^{n-\alpha} f^{(n)}(t),\ t>0. \tag{2.4}$$

The Caputo derivative of constant is equal to zero.

Lemma 2.5 ([21]). *Let $\alpha>0$, then the differential equation*

$$^cD^\alpha h(t) = 0, \tag{2.5}$$

has solutions $h(t)=c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}$ and $I^\alpha D^\alpha h(t) = h(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}$ where $c_i \in R, i=0,1,\ldots,n-1,\ n=[\alpha]+1$.

To investigate the system (1.4)–(1.7), we first consider the associated linear problem and obtain its solution with the help of lemma 2.5 and adopted the methodology of M. Feckan [7].

Lemma 2.6. *Suppose that $\alpha\in(1,2)$ and the function $\sigma:[0,T]\to X$ be continuously differentiable. A function $u(t)$ is a solution of the fractional integral equation:*

$$u(t) = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds - h(u) + \int_0^T q_1(u(s))ds + \Big[\frac{\Gamma(2-q)}{T^{1-q}}\int_0^T q_2(u(s))ds & \\ -\frac{\Gamma(2-q)}{T^{1-q}}\int_0^T \frac{(T-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}\sigma(s)ds - \sum\limits_{i=1}^m \frac{\Gamma(2-q)}{t_i{}^{1-q}} J_i\Big(\sum\limits_{j=1}^i d_{i,j} I_t^{\beta_{i,j}} u(t_j^-)\Big)\Big]t, & t\in[0,t_1), \\ \ldots & \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \sum\limits_{i=1}^k S_i\Big(\sum\limits_{j=1}^i c_{i,j} I_t^{\beta_{i,j}} u(i_j^-)\Big) - h(u) + \int_0^T q_1(u(s))ds & \\ +\Big[\frac{\Gamma(2-q)}{T^{1-q}}\int_0^T q_2(u(s))ds - \frac{\Gamma(2-q)}{T^{1-q}}\int_0^T \frac{(T-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}\sigma(s)ds & \\ -\sum\limits_{i=1}^m \frac{\Gamma(2-q)}{t_i{}^{1-q}} J_i\Big(\sum\limits_{j=1}^i d_{i,j} I_t^{\beta_{i,j}} u(t_j^-)\Big)\Big]t & \\ +\sum\limits_{i=1}^k (t-t_i)\frac{\Gamma(2-q)}{t_i{}^{1-q}} J_i\Big(\sum\limits_{j=1}^i d_{i,j} I_t^{\beta_{i,j}} u(t_j^-)\Big), & t\in(t_k,t_{k+1}]. \end{cases} \tag{2.6}$$

if and only if $u(t)$ is a solution of the following boundary value problem

$$\begin{cases} ^cD^\alpha u(t) = \sigma(t),\ 1<\alpha<2,\ t\in[0,T], \\ \Delta u(t_k) = S_k\Big(\sum\limits_{j=1}^k c_{k,j} I_t^{\beta_{k,j}} u(t_j^-)\Big), \\ \Delta(^cD^q u(t_k)) = J_k\Big(\sum\limits_{j=1}^k d_{k,j} I_t^{\beta_{k,j}} u(t_j^-)\Big),\ 0<q<1, \\ u(0)+h(u) = \int_0^T q_1(u(s))ds, \quad D^q u(0) + D^q u(T) = \int_0^T q_2(u(s))ds. \end{cases} \tag{2.7}$$

Proof. For $t\in[0,t_1)$, let $u(t)$ be the solution of (2.7), we have

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds - c_0 - c_1 t. \tag{2.8}$$

Using the boundary condition $u(0)+h(u) = \int_0^T q_1(u(s))ds$, we compute c_0 and put it into (2.8), we get

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds - h(u) + \int_0^T q_1(u(s))ds - c_1 t. \tag{2.9}$$

For $t\in(t_1,t_2]$, we may write the solution

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds - c_2 - c_3 t, \tag{2.10}$$

by apply impulsive condition $\Delta u(t_1) = S_1\Big(\sum_{j=1}^{1} c_{1,j} I_t^{\beta_{1,j}} u(t_j^-)\Big)$, we compute the value of constant c_2 and insert it into (2.10), we incur

$$\begin{aligned} u(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + S_1\Big(\sum_{j=1}^{1} c_{1,j} I_t^{\beta_{1,j}} u(t_j^-)\Big) + c_3(t_1 - t) \\ &- h(u) + \int_0^T q_1(u(s))ds - c_1 t_1, \end{aligned} \tag{2.11}$$

Using impulsive condition $\Delta(D^q u(t_1)) = J_1\Big(\sum_{j=1}^{1} d_{1,j} I_t^{\beta_{1,j}} u(t_j^-)\Big)$, we compute c_3 and insert it into (2.11), we have

$$\begin{aligned} u(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + S_1\Big(\sum_{j=1}^{1} c_{1,j} I_t^{\beta_{1,j}} u(t_j^-)\Big) - h(u) + \int_0^T q_1(u(s))ds \\ &+ \frac{\Gamma(2-q)}{{t_1}^{1-q}} J_1\Big(\sum_{j=1}^{1} d_{1,j} I_t^{\beta_{1,j}} u(t_j^-)\Big)(t - t_1) - c_1 t. \end{aligned} \tag{2.12}$$

Similarly for $t \in (t_k, t_{k+1}]$, we may write the solution as,

$$\begin{aligned} u(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sigma(s)ds + \sum_{i=1}^{k} S_i\Big(\sum_{j=1}^{i} c_{i,j} I_t^{\beta_{i,j}} u(i_j^-)\Big) - h(u) - c_1 t \\ &+ \sum_{i=1}^{k}(t - t_i)\frac{\Gamma(2-q)}{{t_i}^{1-q}} J_i\Big(\sum_{j=1}^{i} d_{i,j} I_t^{\beta_{i,j}} u(t_j^-)\Big) + \int_0^T q_1(u(s))ds. \end{aligned} \tag{2.13}$$

Use the boundary condition $D^q u(0) + D^q u(T) = \int_0^T q_2(u(s))ds$ in (2.13), we obtain the required value of the constant c_1 as

$$\begin{aligned} c_1 = &\frac{\Gamma(2-q)}{T^{1-q}}\Big[- \int_0^T q_2(u(s))ds + \int_0^T \frac{(T-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}\sigma(s)ds \\ &+ \sum_{i=1}^{m} \frac{T^{1-q}}{\Gamma(2-q)} \frac{\Gamma(2-q)}{{t_i}^{1-q}} J_i\Big(\sum_{j=1}^{i} d_{i,j} I_t^{\beta_{i,j}} u(t_j^-)\Big)\Big]. \end{aligned} \tag{2.14}$$

Conversely, we assume that u satisfies the impulsive fractional integral equation (2.6) then by direct computation it can be seen that the solution given by (2.6) satisfies (2.7). This completes the proof of the lemma. □

3. Main results

Definition 3.1. The function $u : [0,T] \to X$ such that $u \in PC_t^1([0,T] : X)$ is said to be the solution of the system (1.4)–(1.7) iff it satisfied the following integral equation

$$u(t)=\begin{cases}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(f(s,u(s))+\int_0^s g(s-\tau)p(\tau,u(\tau))d\tau)ds-h(u)+\int_0^T q_1(u(s))ds\\ +t\Big[\frac{\Gamma(2-q)}{T^{1-q}}\Big(\int_0^T q_2(u(s))ds-\int_0^T\frac{(T-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}(f(s,u(s))+\int_0^s g(s-\tau)p(\tau,u(\tau))d\tau)ds\Big)\\ -\sum\limits_{i=1}^{m}\frac{\Gamma(2-q)}{t_i{}^{1-q}}J_i\Big(\sum\limits_{j=1}^{i}d_{i,j}I_t^{\beta_{i,j}}u(t_j^-)\Big)\Big], & t\in[0,t_1).\\ \dots\\ \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(f(s,u(s))+\int_0^s g(s-\tau)p(\tau,u(\tau))d\tau)ds+\sum\limits_{i=1}^{k}S_i\Big(\sum\limits_{j=1}^{i}c_{i,j}I_t^{\beta_{i,j}}u(t_j^-)\Big)\\ -h(u)+\int_0^T q_1(u(s))ds+t\Big[\frac{\Gamma(2-q)}{T^{1-q}}\Big(\int_0^T q_2(u(s))ds-\int_0^T\frac{(T-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}\\ (f(s,u(s))+\int_0^s g(s-\tau)p(\tau,u(\tau))d\tau)ds\Big)-\sum\limits_{i=1}^{m}\frac{\Gamma(2-q)}{t_i{}^{1-q}}J_i\Big(\sum\limits_{j=1}^{i}d_{i,j}I_t^{\beta_{i,j}}u(t_j^-)\Big)\Big]\\ +\sum\limits_{i=1}^{k}(t-t_i)\frac{\Gamma(2-q)}{t_i{}^{1-q}}J_i\Big(\sum\limits_{j=1}^{i}d_{i,j}I_t^{\beta_{i,j}}u(t_j^-)\Big), & t\in(t_k,t_{k+1}].\end{cases}\tag{3.1}$$

We introduce the following assumptions to establish our first result:

(H_1) There exists a function $L_1(t)\in L^{\frac{1}{\rho}}([0,T]:X)$ with $\rho\in(0,\alpha-1)$ such that

$$\|f(t,u)-f(t,x)\|_X\le L_1(t)\|u-x\|_X,$$

for each $t\in[0,T]$ and $x,u\in X$.

(H_2) There exists positive constants L_4,L_5,L_6,L_7 and L_8 such that

$$\|q_1(u)-q_1(x)\|_X\le L_4\|u-x\|_X,$$

$$\|q_2(u)-q_2(x)\|_X\le L_5\|u-x\|_X,$$

$$\|S_k(x)-S_k(y)\|_X\le L_6\|x-y\|_X,$$

$$\|J_k(x)-J_k(y)\|_X\le L_7\|x-y\|_X,$$

$$\|h(u)-h(x)\|_X\le L_8\|u-x\|_X,$$

(H_3) The function $p:X\to X$ is continuous and there exists a function $L_2(t)\in L^1([0,T]:X)$ such that

$$\|p(t,u)-p(t,x)\|_X\le L_2(t)\|u-x\|_X,$$

for each $t\in[0,T]$ and $x,u\in X$.

Our first result is based on Banach contraction theorem.

Theorem 3.2. *Suppose that the assumptions $(H_1)-(H_3)$ holds and the following inequality satisfied*

$$\begin{aligned}\Delta\ =\ &\Big[\frac{L^*T^{\alpha-\rho}}{\Gamma(\alpha)}\Big(\frac{1-\rho}{\alpha-\rho}\Big)^{1-\rho}+L_8+L_4T+\Gamma(2-q)L_5T^{1+q}+mL_6\sum_{j=1}^{i}\frac{c_{i,j}t_j^{\beta_{i,j}}}{\Gamma(\beta_{i,j}+1)}\\ &+\frac{L^*T^{\alpha-\rho}\Gamma(2-q)}{\Gamma(\alpha-q)}\Big(\frac{1-\rho}{\alpha-q-\rho}\Big)^{1-\rho}+2mT^q\Gamma(2-q)L_7\sum_{j=1}^{i}\frac{d_{i,j}t_j^{\beta_{i,j}}}{\Gamma(\beta_{i,j}+1)}\Big]<1,\end{aligned}$$

where $L^*=\Big(\int_0^t(L_1(s)+L_2(\tau)G^*)^{\frac{1}{\rho}}ds\Big)^{\rho}$, $G^*=\sup\limits_{t\in[0,T]}\int_0^t g(t-s)ds$. Then the system (1.4)–(1.7) has a unique solution.

Proof. We transform the problem (1.4)–(1.7) into a fixed point problem. Consider an operator $N : PC_t \to PC_t$, defined by

$$(Nu)t = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(f(s,u(s)) + \int_0^s g(s-\tau)p(\tau,u(\tau))d\tau)ds - h(u) \\ + \int_0^T q_1(u(s))ds + t\Big[\frac{\Gamma(2-q)}{T^{1-q}}\Big(\int_0^T q_2(u(s))ds - \int_0^T \frac{(T-s)^{\alpha-q-1}}{\Gamma(\alpha-q)} \\ (f(s,u(s)) + \int_0^s g(s-\tau)p(\tau,u(\tau))d\tau)ds\Big) \\ - \sum\limits_{i=1}^m \frac{\Gamma(2-q)}{{t_i}^{1-q}} J_i\Big(\sum\limits_{j=1}^i d_{i,j} I_t^{\beta_{i,j}} u(t_j^-)\Big)\Big], & t \in [0,t_1), \\ \dots \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(f(s,u(s)) + \int_0^s g(s-\tau)p(\tau,u(\tau))d\tau)ds \\ + \sum\limits_{i=1}^k S_i\Big(\sum\limits_{j=1}^i c_{i,j} I_t^{\beta_{i,j}} u(t_j^-)\Big) - h(u) + \int_0^T q_1(u(s))ds \\ +t\Big[\frac{\Gamma(2-q)}{T^{1-q}}\Big(\int_0^T q_2(u(s))ds - \int_0^T \frac{(T-s)^{\alpha-q-1}}{\Gamma(\alpha-q)} \\ (f(s,u(s)) + \int_0^s g(s-\tau)p(\tau,u(\tau))d\tau)ds\Big) \\ - \sum\limits_{i=1}^m \frac{\Gamma(2-q)}{{t_i}^{1-q}} J_i\Big(\sum\limits_{j=1}^i d_{i,j} I_t^{\beta_{i,j}} u(t_j^-)\Big)\Big] \\ + \sum\limits_{i=1}^k (t-t_i)\frac{\Gamma(2-q)}{{t_i}^{1-q}} J_i\Big(\sum\limits_{j=1}^i d_{i,j} I_t^{\beta_{i,j}} u(t_j^-)\Big), & t \in (t_k, t_{k+1}]. \end{cases} \tag{3.2}$$

To show that N has fixed point on $[0,T]$, consider $u_1, u_2 \in PC_t$ and for $t \in (t_k, t_{k+1}]$, we have

$$\begin{aligned} &\|(Nu_1) - (Nu_2)\|_X \\ &\quad \le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(\|f(s,u_1(s)) - f(s,u_2(s))\|_X \\ &\qquad + \int_0^s g(s-\tau)\|p(\tau,u_1(\tau)) - p(\tau,u_2(\tau))\|_X d\tau\Big)ds + \|h(u_1) - h(u_2)\|_X \\ &\qquad + \sum_{i=1}^k \|S_i\Big(\sum_{j=1}^i c_{i,j} I_t^{\beta_{i,j}} u_1(t_j^-)\Big) - S_i\Big(\sum_{j=1}^i c_{i,j} I_t^{\beta_{i,j}} u_2(t_j^-)\Big)\|_X \\ &\qquad + \int_0^T \|q_1(u_1(s)) - q_1(u_2(s))\|_X ds + \Big[\frac{\Gamma(2-q)}{T^{1-q}}\Big(\int_0^T \|q_2(u_1(s)) - q_2(u_2(s))\|_X ds \\ &\qquad + \int_0^T \frac{(T-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}\Big(\|f(s,u_1(s)) - f(s,u_2(s))\|_X \\ &\qquad + \int_0^s g(s-\tau)\|p(\tau,u_1(\tau)) - p(\tau,u_2(\tau))\|_X d\tau\Big)ds\Big) \\ &\qquad + \sum_{i=1}^m \frac{\Gamma(2-q)}{|t_i|^{1-q}}\|J_i\Big(\sum_{j=1}^i d_{i,j} I_t^{\beta_{i,j}} u_1(t_j^-)\Big) - J_i\Big(\sum_{j=1}^i d_{i,j} I_t^{\beta_{i,j}} u_2(t_j^-)\Big)\|_X\Big]|t| \\ &\qquad + \sum_{i=1}^k (|t-t_i|)\frac{\Gamma(2-q)}{|t_i|^{1-q}}\|J_i\Big(\sum_{j=1}^i d_{i,j} I_t^{\beta_{i,j}} u_1(t_j^-)\Big) - J_i\Big(\sum_{j=1}^i d_{i,j} I_t^{\beta_{i,j}} u_2(t_j^-)\Big)\|_X \\ &\quad \le \frac{1}{\Gamma(\alpha)}\Big(\int_0^t ((t-s)^{\alpha-1})^{\frac{1}{1-\rho}} ds\Big)^{1-\rho}\Big(\int_0^t (L_1(s)\|u_1 - u_2\| + L_2(\tau)G^*\|u_1 - u_2\|)^{\frac{1}{\rho}} ds\Big)^{\rho} \\ &\qquad + \|h(u_1) - h(u_2)\|_X + \sum_{i=1}^k \|S_i\Big(\sum_{j=1}^i c_{i,j} I_t^{\beta_{i,j}} u_1(t_j^-)\Big) - S_i\Big(\sum_{j=1}^i c_{i,j} I_t^{\beta_{i,j}} u_2(t_j^-)\Big)\|_X \end{aligned}$$

$$
\begin{aligned}
&+\int_0^T \|q_1(u_1(s))-q_1(u_2(s))\|_X ds+\Big[\frac{\Gamma(2-q)}{T^{1-q}}\Big(\int_0^T \|q_2(u_1(s))-q_2(u_2(s))\|_X ds\\
&+\frac{1}{\Gamma(\alpha-q)}\Big(\int_0^t((T-s)^{\alpha-q-1})^{\frac{1}{1-\rho}}ds\Big)^{1-\rho}\Big(\int_0^t(L_1(s)\|u_1-u_2\|+L_2(\tau)G^*\|u_1-u_2\|)^{\frac{1}{\rho}}ds\Big)^{\rho}\Big)\\
&+\sum_{i=1}^m\frac{\Gamma(2-q)}{|t_i|^{1-q}}\|J_i\Big(\sum_{j=1}^i d_{i,j}I_t^{\beta_{i,j}}u_1(t_j^-)\Big)-J_i\Big(\sum_{j=1}^i d_{i,j}I_t^{\beta_{i,j}}u_2(t_j^-)\Big)\|_X\Big]|t|\\
&+\sum_{i=1}^k(|t-t_i|)\frac{\Gamma(2-q)}{|t_i|^{1-q}}\|J_i\Big(\sum_{j=1}^i d_{i,j}I_t^{\beta_{i,j}}u_1(t_j^-)\Big)-J_i\Big(\sum_{j=1}^i d_{i,j}I_t^{\beta_{i,j}}u_2(t_j^-)\Big)\|_X.
\end{aligned}
$$

By taking the considered norm and assumptions $(H_1)-(H_3)$, we estimate as

$$
\begin{aligned}
\|(Nu_1)-(Nu_2)\|_{PC_t}\le\Big[&\frac{L^*T^{\alpha-\rho}}{\Gamma(\alpha)}\Big(\frac{1-\rho}{\alpha-\rho}\Big)^{1-\rho}+L_8+L_4T+\Gamma(2-q)L_5T^{1+q}+mL_6\sum_{j=1}^i\frac{c_{i,j}t_j^{\beta_{i,j}}}{\Gamma(\beta_{i,j}+1)}\\
&+\frac{L^*T^{\alpha-\rho}\Gamma(2-q)}{\Gamma(\alpha-q)}\Big(\frac{1-\rho}{\alpha-q-\rho}\Big)^{1-\rho}+2mT^q\Gamma(2-q)L_7\sum_{j=1}^i\frac{d_{i,j}t_j^{\beta_{i,j}}}{\Gamma(\beta_{i,j}+1)}\Big]\|u_1-u_2\|_{PC_t}\\
&\le\Delta\|u_1-u_2\|_{PC_t}.
\end{aligned}
$$

Hence the operator N is a contraction map and has a fixed point $u\in PC_t$. In a consequence of Banach fixed point theorem the system (1.4)–(1.7) has a unique solution on the interval $[0,T]$. This completes the proof of the theorem. □

Second result of the paper based on Krasnoselkii's fixed point theorem [21]. For this, we introduce the following additional assumptions:

(H_4) There exists $\varsigma\in(0,\alpha-1)$ and real functions $M_1(t),M_2(t)\in L^{\frac{1}{\varsigma}}([0,T]:X)$ such that

$$\|f(t,u)\|_X\le M_1(t),$$

$$\|p(t,u)\|_X\le M_2(t),$$

for all $u\in X$.

(H_5) If h,q_1,q_2,S_k,J_k are continuous bounded functions and there exist positive constants M_3,M_4,M_5,M_6,M_7 s.t.

$$\|h(u)\|_X\le M_3,$$

$$\|q_1(u)\|_X\le M_4,$$

$$\|q_2(u)\|_X\le M_5,$$

$$\|S_k(y)\|_X\le M_6,$$

$$\|J_k(y)\|_X\le M_7,$$

for all $u,y\in X$.

Theorem 3.3. *Let f be any continuous function. Assume that (H_1), (H_3), (H_4) and (H_5) holds with the condition $\varrho < 1$, where*

$$\varrho = \frac{L^* T^{\alpha-\rho}}{\Gamma(\alpha)}\Big(\frac{1-\rho}{\alpha-\rho}\Big)^{1-\rho} + \frac{L^* T^{\alpha-\rho}\Gamma(2-q)}{\Gamma(\alpha-q)}\Big(\frac{1-\rho}{\alpha-q-\rho}\Big)^{1-\rho}.$$

Then the problem (1.4)–(1.7) has at least one solution on $[0,T]$.

Proof. Let

$$\begin{aligned} r \geq\ & mM_6 + M_3 + M_4 T + T^{q+1}\Gamma(2-q)M_5 + 2\Gamma(2-q)mT^q M_7 \\ & + \frac{M^* T^{\alpha-\varsigma}}{\Gamma(\alpha)}\Big(\frac{1-\varsigma}{\alpha-\varsigma}\Big)^{1-\varsigma} + \frac{M^* T^{\alpha-\varsigma}\Gamma(2-q)}{\Gamma(\alpha-q)}\Big(\frac{1-\varsigma}{\alpha-q-\varsigma}\Big)^{1-\varsigma}, \end{aligned} \tag{3.3}$$

and $M^* = \Big(\int_0^t (M_1(s) + M_2(\tau))^{\frac{1}{\varsigma}}\Big)^{\varsigma}$. Consider the space $PC_t^r = \{u \in PC_t : \|u\|_{PC_t} \leq r\}$, then PC_t^r is a bounded, closed convex subset in PC_t. Define the operators $Q, P : PC_t^r \to PC_t^r$ as

$$(Qu) = \begin{cases} \sum_{i=1}^{k} S_i\Big(\sum_{j=1}^{i} c_{i,j} I_t^{\beta_{i,j}} u(t_j^-)\Big) - h(u) + \int_0^T q_1(u(s))ds \\ +t\frac{\Gamma(2-q)}{T^{1-q}}\int_0^T q_2(u(s))ds - \sum_{i=1}^{m} t\frac{\Gamma(2-q)}{t_i^{1-q}} J_i\Big(\sum_{j=1}^{i} d_{i,j} I_t^{\beta_{i,j}} u(t_j^-)\Big) \\ +\sum_{i=1}^{k}(t-t_i)\frac{\Gamma(2-q)}{t_i^{1-q}} J_i\Big(\sum_{j=1}^{i} d_{i,j} I_t^{\beta_{i,j}} u(t_j^-)\Big), \end{cases} \tag{3.4}$$

$$(Pu) = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(f(s,u(s)) + \int_0^s g(s-\tau)p(\tau,u(\tau))d\tau)ds \\ -t\frac{\Gamma(2-q)}{T^{1-q}}\int_0^T \frac{(T-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}(f(s,u(s)) + \int_0^s g(s-\tau)p(\tau,u(\tau))d\tau)ds. \end{cases} \tag{3.5}$$

The proof of the theorem 3.3 for $t \in (t_k, t_{k+1}]$ compile in the following steps:

Step 1. We show that $Qu + Pv \in PC_t^r$. Let $u, v \in PC_T^r$, we have

$$\begin{aligned} \|Q(u) + P(v)\|_X \leq\ & \sum_{i=1}^{k} \|S_i\Big(\sum_{j=1}^{i} c_{i,j} I_t^{\beta_{i,j}} u(t_j^-)\Big)\|_X + \|h(u)\|_X + \int_0^T \|q_1(u(s))\|_X ds + |t|\frac{\Gamma(2-q)}{T^{1-q}} \\ & \int_0^T \|q_2(u(s))\|_X ds + \sum_{i=1}^{m}\frac{\Gamma(2-q)|t|}{|t_i|^{1-q}}\|J_i\Big(\sum_{j=1}^{i} d_{i,j} I_t^{\beta_{i,j}} u(t_j^-)\Big)\|_X \\ & + \sum_{i=1}^{k} |(t-t_i)|\frac{\Gamma(2-q)}{|t_i|^{1-q}}\|J_i\Big(\sum_{j=1}^{i} d_{i,j} I_t^{\beta_{i,j}} u(t_j^-)\Big)\|_X \\ & + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|(f(s,v(s)) + \int_0^s g(s-\tau)p(\tau,v(\tau))d\tau)\|_X ds \\ & + |t|\frac{\Gamma(2-q)}{T^{1-q}}\int_0^T \frac{(T-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}\|(f(s,v(s)) + \int_0^s g(s-\tau)p(\tau,v(\tau))d\tau)\|_X ds. \end{aligned}$$

By using the assumptions $(H_4) - (H_5)$, we get

$$\begin{aligned} \|Q(u) + P(v)\|_{PC_t} \leq\ & \Big[mM_6 + M_3 + M_4 T + T^{q+1}\Gamma(2-q)M_5 + 2\Gamma(2-q)mT^q M_7 \\ & + \frac{M^* T^{\alpha-\varsigma}}{\Gamma(\alpha)}\Big(\frac{1-\varsigma}{\alpha-\varsigma}\Big)^{1-\varsigma} + \frac{M^* T^{\alpha-\varsigma}\Gamma(2-q)}{\Gamma(\alpha-q)}\Big(\frac{1-\varsigma}{\alpha-q-\varsigma}\Big)^{1-\varsigma}\Big] \leq r. \end{aligned}$$

Since $\|Qu + Pv\|_{PC_t} \leq r$, we have $Qu + Pv \in PC_t^r$ which shows that PC_t^r is closed with respect to both the operators.

Step 2. Let $u_n \to u$ be any convergent sequence in PC_t^r, we have

$$\begin{aligned}\|Q(u_n)-Q(u)\|_X \le& \sum_{i=1}^{k}\|S_i\Big(\sum_{j=1}^{i}c_{i,j}I_t^{\beta_{i,j}}u_n(t_j^-)\Big)-S_i\Big(\sum_{j=1}^{i}c_{i,j}I_t^{\beta_{i,j}}u(t_j^-)\Big)\|_X+\|h(u_n)-h(u)\|_X\\&+\int_0^T\|q_1(u_n(s))-q_1(u(s))\|_X ds+|t|\frac{\Gamma(2-q)}{T^{1-q}}\int_0^T\|q_2(u_n(s))-q_2(u(s))\|_X ds\\&+\sum_{i=1}^{m}\frac{\Gamma(2-q)|t|}{|t_i|^{1-q}}\|J_i\Big(\sum_{j=1}^{i}d_{i,j}I_t^{\beta_{i,j}}u_n(t_j^-)\Big)-J_i\Big(\sum_{j=1}^{i}d_{i,j}I_t^{\beta_{i,j}}u(t_j^-)\Big)\|_X\\&+\sum_{i=1}^{k}|(t-t_i)|\frac{\Gamma(2-q)}{|t_i|^{1-q}}\|J_i\Big(\sum_{j=1}^{i}d_{i,j}I_t^{\beta_{i,j}}u_n(t_j^-)\Big)-J_i\Big(\sum_{j=1}^{i}d_{i,j}I_t^{\beta_{i,j}}u(t_j^-)\Big)\|_X. \quad (3.6)\end{aligned}$$

Since the functions $h, q_1, q_2, I_k, J_k,\ k = 1, 2, \dots, m$, are continuous, hence from (3.6), we may conclude that $\|(Qu_n) - (Qu)\|_{PC_t^r} \to 0$ which implies that the mapping Q is continuous on PC_t^r.

Step 3. To show that Q is uniformly bounded, let $u \in PC_t^r$, we have

$$\begin{aligned}\|Q(u)\|_X \le& \sum_{i=1}^{k}\|S_i\Big(\sum_{j=1}^{i}c_{i,j}I_t^{\beta_{i,j}}u(t_j^-)\Big)\|_X+\|h(u)\|_X+\int_0^T\|q_1(u(s))\|_X ds\\&+|t|\frac{\Gamma(2-q)}{T^{1-q}}\int_0^T\|q_2(u(s))\|_X ds+\sum_{i=1}^{m}\frac{\Gamma(2-q)|t|}{|t_i|^{1-q}}\|J_i\Big(\sum_{j=1}^{i}d_{i,j}I_t^{\beta_{i,j}}u(t_j^-)\Big)\|_X\\&+\sum_{i=1}^{k}|(t-t_i)|\frac{\Gamma(2-q)}{|t_i|^{1-q}}\|J_i\Big(\sum_{j=1}^{i}d_{i,j}I_t^{\beta_{i,j}}u(t_j^-)\Big)\|_X.\end{aligned}$$

Combining the assumptions $(H_4) - (H_5)$, we estimate as

$$\|Q(u)\|_{PC_t} \le mM_6 + M_3 + M_4T + T^{1+q}\Gamma(2-q)M_5 + 2\Gamma(2-q)mM_7T^q.$$

Hence the mapping Q is uniformly bounded in PC_t^r.

Step 4. Let $l_1, l_2 \in [0, T],\ t_k \le l_1 < l_2 \le t_{k+1},\ k = 1, 2, \dots, m,\ u \in PC_t^r$, we have

$$\begin{aligned}\|Q(u)(l_2)-Q(u)(l_1)\|_X \le& (l_2-l_1)\frac{\Gamma(2-q)}{T^{1-q}}\int_0^T\|q_2(u(s))\|_X ds+\sum_{i=1}^{m}\frac{\Gamma(2-q)(l_2-l_1)}{|t_i|^{1-q}}\\&\|J_i\Big(\sum_{j=1}^{i}d_{i,j}I_t^{\beta_{i,j}}u(t_j^-)\Big)\|_X+\sum_{i=1}^{k}\frac{\Gamma(2-q)(l_2-l_1)}{|t_i|^{1-q}}\|J_i\Big(\sum_{j=1}^{i}d_{i,j}I_t^{\beta_{i,j}}u(t_j^-)\Big)\|_X.\end{aligned}$$

As $l_2 \to l_1$, we may conclude that $\|Q(u)(l_2) - Q(u)(l_1)\|_{PC_t^r} \to 0$. Which show that Q, is equi-continuous mapping in $(t_k, t_{k+1}]$. Combing **Step 2** to **Step 4** together with the Arzela-Ascoli's theorem (taken from the paper [21] theorem 2.12, p.p–3012), we conclude that the operator Q is a compact map on PC_t^r.

Step 5. Let $u, u^* \in PC_t^r,\ t \in [0, T],\ k = 1, 2, \dots, m$, we have

$$\begin{aligned}\|P(u)-P(u^*)\|_X \le& \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Big(\|f(s,u(s))-f(s,u^*(s))\|_X\\&+\int_0^s g(s-\tau)\|p(\tau,u(\tau))-p(\tau,u^*(\tau))\|_X d\tau\Big)ds\\&+|t|\frac{\Gamma(2-q)}{T^{1-q}}\int_0^T\frac{(T-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}\Big(\|f(s,u(s))-f(s,u^*(s))\|_X\\&+\int_0^s g(s-\tau)\|p(\tau,u(\tau))-p(\tau,u^*(\tau))\|_X d\tau\Big)ds.\end{aligned}$$

Using (H_1) and (H_3), we estimate as

$$\begin{aligned}\|P(u)-P(u^*)\|_{PC_t} &\leq \Big[\frac{L^*T^{\alpha-\rho}}{\Gamma(\alpha)}\Big(\frac{1-\rho}{\alpha-\rho}\Big)^{1-\rho}+\frac{L^*T^{\alpha-\rho}\Gamma(2-q)}{\Gamma(\alpha-q)}\Big(\frac{1-\rho}{\alpha-q-\rho}\Big)^{1-\rho}\Big]\|u-u^*\|_{PC_t}\\ &\leq \varrho\|u-u^*\|_{PC_t}.\end{aligned}$$

Hence, we have $\|(Pu)-(Pu^*)\|_{PC_t}\leq \varrho\|u-u^*\|_{PC_t}$ implies that the operator P is a contraction mapping on PC_t^r. Thus all the assumptions of the Krasnoselkii's fixed point theorem satisfied. Which implies that the set PC_t^r has a fixed point which is the solution of system (1.4)–(1.7) on $[0,T]$. This completes the proof of the theorem. □

4. Example

Example 4.1. Consider the following fractional order impulsive integro-differential equation with nonlocal conditions:

$$\begin{cases} {}^cD^{3/2}u(t)=\frac{|u|}{(22+23e^t)(1+|u|)}+\int_0^t\frac{e^{(t-s)}}{45}|u(s)|ds,\ t\in[0,1],\ t\neq(1/3),\\ \Delta u(1/3)=I_1\Big(\frac{3}{47}\int_0^{1/3}\frac{(t-s)^{-2/3}}{\Gamma(1/3)}u(s)ds\Big),\\ \Delta({}^cD^{1/2}u(1/3))=J_1\Big(\frac{5}{49}\int_0^{1/3}\frac{(t-s)^{-2/3}}{\Gamma(1/3)}u(s)ds\Big),\\ u(0)+\sum\limits_{i=1}^n c_iu(t_i)=\int_0^1\frac{|u(s)|}{20+|u(s)|}ds,\ D^{\frac12}u(0)+D^{\frac12}u(1)=\int_0^1\frac{|u(s)|}{23+|u(s)|}ds,\end{cases} \tag{4.1}$$

where $I_1(u)=\frac{4|u|}{(6+5|u|)}$, $J_1(u)=\frac{3|u|}{(5+4|u|)}$ and $0<t_1<\cdots<t_n<1$, c_i, $i=1,\ldots,n$ are given positive constants with $\sum\limits_{i=1}^n c_i<\frac18$. Furthermore, we have $f(t,u(t))+\int_0^t g(t-s)p(t,u(s))ds=\frac{|u|}{(22+23e^t)(1+|u|)}+\int_0^t\frac{e^{(t-s)}}{45}|u(s)|ds$. Let $u,u'\in X$ then for $t\in[0,1]$, we may verify the assumptions of theorem 3.2 as follows

$$\begin{aligned}\|f(t,u)-f(t,v)\|_X &= \frac{1}{(22+23e^t)}\Big\|\frac{|u|}{1+|u|}-\frac{|v|}{1+|v|}\Big\|_X\\ &= \frac{1}{(22+23e^t)}\Big\|\frac{|u-v|}{(1+|u|)(1+|v|)}\Big\|_X\\ &\leq \frac{1}{(22+23e^t)}\|u-v\|_X\\ &\leq \frac{1}{45}\|u-v\|_X.\end{aligned}$$

$$\int_0^t g(t-s)\|p(t,u(s))-p(t,v(s))\|_X ds=\int_0^t\frac{e^{(t-s)}}{45}\|u(s)-v(s)\|_X ds\leq\frac{1}{45}\|u(s)-v(s)\|_X.$$

$$\|q_1(u)-q_1(x)\|_X\leq\frac{1}{20}\|u-x\|_X,$$

$$\|q_2(u)-q_2(x)\|_X\leq\frac{1}{23}\|u-x\|_X,$$

for all $u,x\in X$.

$$\|I_1(x)-I_1(y)\|_X\leq\frac23\|x-y\|_X,$$

$$\|J_1(x) - J_1(y)\|_X \le \frac{3}{5}\|x - y\|_X,$$

for all $x, y \in X$.

$$\|h(u) - h(x)\|_X \le \sum_{i=1}^{n} c_i\|u - x\|_X \le \frac{1}{8}\|u - x\|_X, \ \forall\, u, x \in X.$$

Here $q = \frac{1}{2},\ \alpha = \frac{3}{2},\ T = 1,\ \beta_{1,1} = \frac{1}{2},\ c_{1,1} = \frac{3}{47},\ d_{1,1} = \frac{5}{49},\ L_4 = \frac{1}{20},\ L_5 = \frac{1}{23},\ L_6 = \frac{2}{3},\ L_7 = \frac{3}{5},\ L_8 = \frac{1}{8}$. Obviously, $L(t) \in L^4([0,1], R^+),\ \rho = \frac{1}{4}$ and

$$L^* = \Big(\int_0^t (L_1(s) + L_2(\tau)G^*)^{\frac{1}{\rho}} ds\Big)^{\rho} = \Big(\int_0^1 (\frac{2}{45})^4 ds\Big)^{\frac{1}{4}} = \frac{2}{45}.$$

As we know that $\Gamma(\frac{1}{2}) \approx \sqrt{\pi},\ \Gamma(\frac{3}{2}) \approx .8862,\ \Gamma(\frac{5}{2}) \approx 1.33$, we have

$$\begin{aligned}\Delta = \Big[&\frac{\frac{2}{45}}{\Gamma(\frac{3}{2})}\Big(\frac{1-\frac{1}{4}}{\frac{3}{2}-\frac{1}{4}}\Big)^{1-\frac{1}{4}} + \frac{1}{8} + \frac{1}{20} + \Gamma(2-\frac{1}{2})\frac{1}{23} + \frac{2}{3}\frac{\frac{3}{47}}{\Gamma(\frac{1}{2}+1)}\\ &+ \frac{\frac{2}{45}\Gamma(2-\frac{1}{2})}{\Gamma(\frac{3}{2}-\frac{1}{2})}\Big(\frac{1-\frac{1}{4}}{\frac{3}{2}-\frac{1}{2}-\frac{1}{4}}\Big)^{1-\frac{1}{4}} + 2\Gamma(2-\frac{1}{2})\frac{3}{5}\frac{\frac{5}{49}}{\Gamma(\frac{1}{2}+1)}\Big]\\ =\ & 0.45757.\end{aligned}$$

Since $\Delta = 0.45757 < 1$. Therefore, by theorem 3.2 the system (4.1) has a unique solution on $[0,1]$.

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