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The subspace of G-invariant vectors in isometric representations

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Abstract

Among other things it is shown that, in a dual isometric representation of a group on a rotund and smooth dual Banach space, the space of invariant vectors is 1-complemented.

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1. Introduction

Given a group G and a vector space X over a division ring K, a representation of G in X is a group homomorphism $\pi : G \to \mathsf{GL}(X)$, where $\mathsf{GL}(X)$ denotes the General Linear Group of X, that is, the group of all K-vector space isomorphisms of X. If X is a real or complex topological vector space, then an isomorphic representation of G in X is a group homomorphism $\pi : G \to \mathrm{Iso}(X)$, where $\mathrm{Iso}(X)$ stands for the group of all (topological) isomorphisms of X. If X is a real or complex normed space, then an isometric representation of G in X is a group homomorphism $\pi : G \to \mathcal{G}_X$, where $\mathcal{G}_X$ denotes the group of surjective linear isometries of X.

In isomorphic or isometric representations of topological groups some continuity of the representation may usually be considered (except for discrete groups). This is mainly the case of unitary representations, that is, isometric representations of a topological group on a Hilbert space which are strongly continuous, that is, continuous when the group of surjective linear isometries on the Hilbert space is endowed with the pointwise convergence topology. We refer the reader to [1, 2, 4, 12] for a wider perspective on all of these concepts.

It is worth mentioning that no continuity assumptions on the representations will affect our results, this is why neither we will restrict to topological groups nor we will consider continuous representations.

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In this paper we will mostly work with isometric representations $\pi : G \to \mathcal{G}_X$ of a (non-necessarily topological) group G on a real or complex normed space X. The main result in this manuscrip (see Corollary 3.9), which constitutes a generalization of [1, Proposition 2.6], follows to conclude this introduction.

Theorem 1.1. *Let $\pi : G \to \mathcal{G}_X$ be an isometric representation of the group G on the real or complex normed space X. Then:*

- *If both X and X^* are smooth, then the subspace of invariant vectors of π is quasi-complemented in X.*
- *If both X^* and X^{**} are smooth, then the subspace of invariant vectors of π^* is 1-complemented in X^*.*

2. Preliminaries

This kind of introductory section will be devoted to recall the main geometrical tools upon which we will base most of our results on representation theory. We will begin by recalling basic yet powerful concepts on the theory of topological vector spaces to end up introducing some tools from the geometry of real or complex normed spaces.

2.1. Dual maps and w^*-convergence

Dual maps and w^*-convergence will turn out to be fundamental tools in the development of the further sections. If $T : X \to Y$ is an R-linear mapping between (left) modules X and Y over an associative ring R with unity, then the dual map of T is defined by

$$\begin{array}{rrcl} T^* : & Y^* & \to & X^* \\ & y^* & \mapsto & T^*(y^*) := y^* \circ T. \end{array}$$

Basic properties of dual maps follow:

- $(T \circ S)^* = S^* \circ T^*$.
- If T is invertible, then so is T^* and $(T^*)^{-1} = \left(T^{-1}\right)^*$.
- $T^{**}|_X = T$.

Now recall that if R is a topological ring, then a net $(x_i^*)_{i\in I}$ in X^* is said to be w^*-convergent to $x^* \in X^*$ provided that $(x_i^*(x))_{i\in I}$ converges to $x^*(x)$ for all $x \in X$. It is immediate that if R is a topological ring, then T^* is always w^*-w^* continuous. Conversely, if $R = \mathbb{R}$ or $\mathbb{C}$ and $S : Y^* \to X^*$ is w^*-w^* continuous, then there exists $T : X \to Y$ such that $T^* = S$. Assume now that X and Y are real or complex topological vector spaces. Observe that:

- If T is continuous, then T^* is also continuous.
- If T is an isomorphism, then T^* is also an isomorphism.
- If X and Y are normable and T is an isometry, then T^* is also an isometry.

In a similar way the w-convergence can be defined on X as the inherited w^*-convergence of X from X^{**}, that is a net $(x_i)_{i\in I}$ in X is said to be w-convergent to $x \in X$ provided that $(x^*(x_i))_{i\in I}$ converges to $x^*(x)$ for all $x^* \in X^*$. It is a very simple exercise to check that every vector subspace of a vector space over a topological division ring is always w-closed. Submodules are w-closed provided that they are linearly complemented.

2.2. Annihilators

We are also in need of reminding the reader about the concept of annihilator. Consider X to be a (left) module over an associative ring R with unity. Consider a non-empty subset M of X. The annihilator of M is defined as

$$M^\perp := \{x^* \in X^* : M \subseteq \ker(x^*)\}.$$

It is immediate to realize that:

- $M^{\perp}$ is a submodule of X^* which is w^*-closed in case R is a topological ring.
- If $N \subseteq M \subseteq X$, then $M^{\perp} \subseteq N^{\perp}$.
- If $N, M \subseteq X$, then $(M+N)^{\perp} = M^{\perp} \cap N^{\perp}$.
- If $N, M \subseteq X$, then $(M \cap N)^{\perp} \supseteq M^{\perp} + N^{\perp}$.

If $N \subseteq X^*$ is non-empty, then the pre-annihilator of N is defined by

$$N^{\top} := N^{\perp} \cap X = \bigcap_{n^* \in N} \ker(n^*).$$

It is fairly obvious that $N^{\top}$ is a submodule of X which is w-closed in case R is a topological ring. The following lemma can probably be found in any basic text of Banach spaces or topological vector spaces (see [3]). However we prefer to include the proof in this manuscript for the sake of completeness.

Lemma 2.1. *Let X be a Hausdorff locally convex real or complex topological vector space. Let M and N be vector subspaces of X and X^*, respectively. Then:*

1. $\overline{M} = \left(M^{\perp}\right)^{\perp} \cap X = \left(M^{\perp}\right)^{\top}$.
2. $\overline{N}^{w^*} = \left(N^{\perp} \cap X\right)^{\perp} = \left(N^{\top}\right)^{\perp}$.

Proof.

1. First off, observe that $M \subseteq \left(M^{\perp}\right)^{\perp} \cap X$ and thus $\overline{M} \subseteq \left(M^{\perp}\right)^{\perp} \cap X$. Next, assume to the contrary tha there exists $x \in \left(\left(M^{\perp}\right)^{\perp} \cap X\right) \setminus \overline{M}$. Since X is a Hausdorff locally convex topological vector space, the Hahn-Banach Separation Theorem applies to allow the existence of a continuous linear functional $x^* \in X^*$ such that Re $x^*(x) > \sup \mathrm{Re}\ x^*(M)$, which automatically means that $x^*(M) = \{0\}$, that is, $M \subseteq \ker(x^*)$ and $x^* \in M^{\perp}$. However, $\mathrm{Re} x^*(x) > 0$ and this contradicts the fact that $x \in \left(\left(M^{\perp}\right)^{\perp} \cap X\right)$.
2. Observe again that $N \subseteq \left(N^{\perp} \cap X\right)^{\perp}$. Again, assume to the contrary that there exists $x^* \in \left(N^{\perp} \cap X\right)^{\perp} \setminus \overline{N}^{w^*}$. Since X^* endowed with the w^*-topology is a Hausdorff locally convex topological vector space, the Hahn-Banach Separation Theorem applies to allow the existence of a w^*-continuous linear functional $x \in X$ such that Re $x(x^*) > \sup \mathrm{Re}\ x(N)$, which automatically means that $x(N) = \{0\}$, that is, $N \subseteq \ker(x)$ and $x \in N^{\perp} \cap X$. However, Re $x^*(x) > 0$ and this contradicts the fact that $x^* \in \left(N^{\perp} \cap X\right)^{\perp}$.

□

The reader may notice that the previous lemma strongly relies on the very well known Hahn-Banach Separation Theorem for Hausdorff locally convex real or complex topological vector spaces. However, there is a particular case of (2) of Lemma 2.1 which does not use the Hahn-Banach Separation Theorem and thus it works for vector spaces over Hausdorff topological division rings. Again we include the proof for completeness sake.

Lemma 2.2. *Let X be a vector space over a Hausdorff topological division ring K. A vector subspace A of X^* is w^*-dense if and only if $A^{\top} = \{0\}$.*

Proof. Assume first that we can find $x \in A^{\top} \setminus \{0\}$. Let $x^* \in X^*$ be such that $x^*(x) = 1$. By the w^*-density of A is easy to find $a^* \in A$ such that $a^*(x)$ is sufficiently close to $x^*(x)$ in such a way that $a^*(x) \neq 0$. This contraditcts the fact that $x \in A^{\top}$. Conversely, suppose that $A^{\top} = \{0\}$. Let

$$\mathsf{B}_{X^*}(x^*; x_1, \ldots, x_n; \varepsilon) := \{y^* \in X^* : |y^*(x_i) - x^*(x_i)| \leq \varepsilon \text{ for } 1 \leq i \leq n\}$$

be a w^*-neighborhood of $x^* \in X^*$, where $\{x_1, \ldots, x_n\}$ is a linearly independent subset of X and $\varepsilon > 0$. For every $i \in \{1, \ldots, n\}$ there exists $a_i^* \in A$ such that $a_i^*(x_i) \neq 0$. Because A is a vector subspace of X^* it is possible to construct from $\{a_1^*, \ldots, a_n^*\}$ a linearly independent subset $\{b_1^*, \ldots, b_n^*\} \subset A$ such that $b_i^*(x_j) = \delta_{ij}$ for $1 \leq i, j \leq n$. Finally,

$$b^* := \sum_{i=1}^{n} x^*(x_i)\, b_i^* \in \mathsf{B}_{X^*}(x^*; x_1, \ldots, x_n; \varepsilon) \cap A.$$

□

2.3. Rotundity and smoothness

Recall that a point x in the unit sphere S_X of a real or complex normed space X is said to be an smooth point of B_X provided that there is only one functional in S_{X^*} attaining its norm at x. This unique functional is usually denoted by $\mathsf{J}_X(x)$. The set of smooth points of the (closed) unit ball B_X of X is usually denoted as $\operatorname{smo}(\mathsf{B}_X)$. This way X is said to be smooth provided that $\mathsf{S}_X = \operatorname{smo}(\mathsf{B}_X)$. In case X is smooth, then the dual map of X is defined as the map $\mathsf{J}_X : X \to X^*$ such that $\|\mathsf{J}_X(x)\| = \|x\|$ and $\mathsf{J}_X(x)(x) = \|x\|^2$ for all $x \in X$. It is well known that the dual map is $\|\cdot\|$-w^* continuous and verifies that $\mathsf{J}_X(\lambda x) = \overline{\lambda}\mathsf{J}_X(x)$ for all $\lambda \in \mathbb{C}$ and all $x \in X$. We refer the reader to [6, 8] for a better perspective on these concepts. On the other hand, recall that a normed space is said to be rotund (or strictly convex) provided that its unit sphere is free of non-trivial segments. It is well known among Banach Space Geometers that smoothness and rotundity are dual concepts in the following sense: If a dual space is rotund (smooth), then predual is smooth (rotund). The converse does not hold though. Next, we will gather some of the most relevant results in terms of rotund and smooth renormings into the following one (see [5, Theorem 1 (VII.4)] and [6, Corollary 4.3]).

Theorem 2.3 ([5, 6]). *Let X be a real or complex normed space. Then:*

- *If X is separable, then X admits an equivalent norm so that both X and X^* are rotund.*
- *If X is reflexive, then X admits an equivalent norm so that X is rotund and smooth.*

We will finish this subsection on rotundity and smoothness with a brief introduction on faces and the impact of surjective linear isometries on them. The following definition is very well known amid the Banach Space Geometers.

Definition 2.4. Let X be a real or complex normed space and consider a non-empty convex subset C of B_X. Then:

- C is said to be a face of B_X provided that C verifies the extremal condition with respect to B_X, that is, if $x, y \in \mathsf{B}_X$ and $t \in (0,1)$ are so that $tx + (1-t)\,y \in C$, then $x, y \in C$.
- C is said to be an exposed face of B_X provided that there exists $f \in \mathsf{S}_{X^*}$ such that $C = \mathsf{C}_f$, where $\mathsf{C}_f := f^{-1}(1) \cap \mathsf{B}_X$.

It is inmediate to observe that every exposed face is a proper face, and every proper face must be contained in the unit sphere. If X is a dual space, then C is said to be a w^*-exposed face of B_X provided that the functional that supports or exposes C on B_X is w^*-continuous, that is, an element of the predual of X. When C is a singleton, then we will call it an extreme point, an exposed point, or a w^*-exposed point respectively. Observe that a point $x \in \operatorname{smo}(\mathsf{B}_X)$ if and only if C_x is a singleton (in this situation, $\mathsf{C}_x = \{\mathsf{J}_X(x)\}$). Assume now that $T : X \to Y$ is a surjective linear isometry between the real or complex normed spaces X and Y. It is not difficult to check that if $f \in \mathsf{S}_{X^*}$, then

$$T(\mathsf{C}_f) = \mathsf{C}_{(T^{-1})^*(f)}. \tag{2.1}$$

The fact remarked in the equation right above will play a fundamental role in Lemma 3.7.

2.4. Weaknings of reflexivity

Other geometrical concepts we will be working with are related to reflexivity such as the quasi-reflexivity, the almost-reflexivity, and the dense-reflexivity. The first two concepts are very well known among Banach space theorists, however the last two have been recently introduced for the first time in [7].

Definition 2.5 (García-Pacheco, [7])**.** A real or complex normed space is said to be

- almost-reflexive provided that every continuous linear functional on it is norm-attaining;
- dense-reflexive provided that it is dense in its bidual.

Every almost-reflexive space is dense-reflexive and every reflexive space is almost-reflexive but the converse to those assertions do not hold unless completeness is involved (see [7]). The first example of an almost reflexive space was provided by James in [10], however such spaces are properly defined for the first time in the literature of Banach Space Theory in [7]. The next (original) result shows that other properties of geometric nature also force almost-reflexivity to turn into reflexivity.

Theorem 2.6. *If X is an almost-reflexive real or complex normed space whose dual X^* is smooth, then X is reflexive and rotund.*

Proof. If X is almost-reflexive, then its completion Y verifies that $\mathsf{NA}(Y) = Y^*$, where $\mathsf{NA}(Y)$ denotes the set of continous linear functionals on Y which are norm-attaining. Thus, Y is reflexive and so is $Y^* = X^*$. As a consequence, Y is rotund so [7, Theorem 3.2] applies to deduce that $Y = X$. □

We will finish this subsection by recalling the reader about the concept of quasi-reflexivity (see [9]). A normed space is said to be quasi-reflexive provided that it is of codimension 1 in its bidual. The first example was given by James in [9] where he constructed a separable Banach space X isometrically isomorphic to X^{**} but of codimension 1 in X^{**}. It is obvious that a non-complete quasi-reflexive normed space must be dense-reflexive (the converse is not true either). To summarize we have the following schematic diagram.

UNDER COMPLETENESS

$$\text{Reflexivity} \Leftrightarrow \text{Almost-reflexivity} \Leftrightarrow \text{Dense-reflexivity} \begin{smallmatrix}\not\Rightarrow \\ \not\Leftarrow\end{smallmatrix} \text{Quasi-reflexivity}$$

WITHOUT COMPLETENESS

$$\text{Reflexivity} \begin{smallmatrix}\Rightarrow \\ \not\Leftarrow\end{smallmatrix} \text{Almost-reflexivity} \begin{smallmatrix}\Rightarrow \\ \not\Leftarrow\end{smallmatrix} \text{Dense-reflexivity} \begin{smallmatrix}\not\Rightarrow \\ \Leftarrow\end{smallmatrix} \text{Quasi-reflexivity}$$

2.5. Complementation and quasi-complementation

A submodule Y of a (left) module X over an associative ring R with unity is said to be complemented in X provided that there exists a linear projection from X onto Y, that is, a linear map $P : X \to X$ such that $P^2 = P$ and $P(X) = Y$. In case R is a topological ring and X is a topological module over R, then "complemented" will refer to "topologically complemented", that is, the linear projection must be continuous. The following theorem will play a crucial role in the further sections.

Theorem 2.7. *Let X be a real or complex normed space. Assume that V and W are vector subspaces of X verifying the following condition: there exists $K > 0$ such that for all $v \in V$ and $w \in W$ we have that $\|v\| \leq K\|v + w\|$. Then:*

1. *V and W are K-complemented in $V + W$, that is, there exists a linear projection $P : V + W \to V$ such that* $\ker(P) = W$ *and $\|P\| \leq K$.*
2. *If X is a dual space and both V and W are w^*-closed in X, then $V + W$ is closed in X.*

3. *If X is a dual space with barrelled predual and both V and W are w^*-closed in X, then $V+W$ is sequentially w^*-closed in X.*

Proof.

1. It suffices to consider the following linear projection:
$$\begin{array}{rrcl} P: & V+W & \to & V \\ & v+w & \mapsto & P(v+w) := v. \end{array}$$
Note that the above linear projection is well defined because $V \cap W = \{0\}$. Indeed, let $v \in (V \cap W) \setminus \{0\}$ and consider a real number
$$a \in \left(\frac{-1-K}{K}, \frac{1-K}{K} \right).$$
With this choice of a we have that
$$|1+a| < \frac{1}{K},$$
however
$$\|v\| \le K \|v+av\| = K |1+a| \|v\|,$$
which implies the contradiction that
$$\frac{1}{K} \le |1+a|.$$
2. Let $(v_n + w_n)_{n\in\mathbb{N}}$ be a sequence in $V+W$ which is converging to an element $x \in X$. By hypothesis, we have that $\|v_n\| \le K \|v_n + w_n\|$ for all $n \in \mathbb{N}$, therefore (v_n) is a bounded sequence in V. Since V is w^*-closed, we have that there exists a subnet $(v_{n_i})_{i\in I}$ of $(v_n)_{n\in\mathbb{N}}$ which is w^*-convergent to an element $v \in V$. Notice then that $(w_{n_i})_{i\in I}$ must be w^*-convergent to $x-v$, which means that $x-v \in W$ because of the w^*-closedness of W. Finally, $x = v + (x-v) \in V+W$. Thus $V+W$ is closed.
3. Let $(v_n + w_n)_{n\in\mathbb{N}}$ be a sequence in $V+W$ which is w^*-converging to an element $x \in X$. Notice that $(v_n + w_n)_{n\in\mathbb{N}}$ is bounded because the predual of X is barrelled. Now the proof follows as in item (2) right above.

□

The reader may observe that the proof of (2) of Theorem 2.7 cannot be adapted to further show that $V+W$ be w^*-closed in X due to the fact that w^*-convergent nets need not to be bounded even if the predual is barrelled. On the other hand, two submodules M and N of a topological module are said to be quasi-complemented provided that $M \cap N = \{0\}$ and $\overline{M+N} = X$. Quasi-complementation plays a fundamental role in longstanding open problems such us the famous Separable Quotient Problem (see [11]).

3. The subspace of G-invariant vectors

This section is devoted to entail a complete study on the subspace of G-invariant vectors. We will start off with general representations to end up with isometric representations.

3.1. Dual representations

Dual representations will be crucial in order to study the subspace of G-invariant vectors. Let X be a vector space over a division ring K and consider a representation $\pi : G \to \mathsf{GL}(X)$ of the group G. The dual representation of π is defined as
$$\begin{array}{rrcl} \pi^*: & G & \to & \mathsf{GL}(X^*) \\ & g & \mapsto & \pi^*(g) := \pi\left(g^{-1}\right)^*. \end{array}$$
Keep in mind that if K is a topological division ring, then $\pi^*(G) \subseteq \mathsf{GL}_{w^*}(X^*)$, where $\mathsf{GL}_{w^*}(X^*)$ stands for the w^*-General Linear Group of X^*, that is, the group of all w^*-w^* continuous K-vector space isomorphisms of X^*. On the other hand, also notice the following:

- If $\pi : G \to \mathrm{Iso}(X)$ is an isomorphic representation of the group G on the real or complex topological vector space X, then $\pi^* : G \to \mathrm{Iso}_{w^*}(X^*)$ is also an isomorphic representation of G on X^* endowed with the w^*-topology.
- If $\pi : G \to \mathcal{G}_X$ is an isometric representation of the group G on the real or complex normed space X, then $\pi^* : G \to \mathcal{G}_{w^*}(X^*)$ is also an isometric representation of G on X^*.

Sometimes we will talk directly about "dual representations", that is, representations π of a group G on a vector space X for which there exists a vector space Y and a representation ρ of G on Y such that $X = Y^*$ and $\pi = \rho^*$.

3.2. G-invariant vectors

Let X be a vector space over a division ring K and consider a representation $\pi : G \to \mathsf{GL}(X)$ of the group G.

- A vector subspace Y of X is said to be G-invariant provided that $\pi(g)(Y) = Y$ for all $g \in G$.
- A vector $x \in X$ is said to be G-invariant or G-fixed provided that Kx is G-invariant.
- The subspace of G-invariant vectors is usually denoted by X^G.

The following technical propositions will serve as important tools for the upcoming subsections.

Proposition 3.1. *Let X be a vector space over a division ring K and consider a representation $\pi : G \to \mathsf{GL}(X)$ of the group G. Then:*

1. *X^G is a G-invariant vector subspace of X.*
2. *$(X^*)^G$ is a G-invariant w^*-closed vector subspace of X^* provided that K is a topological division ring.*
3. *$(X^{**})^G \cap X = X^G$.*
4. *If $i : X^G \hookrightarrow X$ is the inclusion map, then $i^{**}\left(\left(X^G\right)^{**}\right) = (X^{**})^G$.*
5. *If K is topological, then $\overline{X^G}^{w^*} = (X^{**})^G$.*

Proof. Only the last two items will be shown.

(4) Let us prove first that $i^{**}\left(\left(X^G\right)^{**}\right) \subseteq (X^{**})^G$. Let $y^{**} \in \left(X^G\right)^{**}$ and $g \in G$. All we need to show is that $\pi^{**}(g)(i^{**}(y^{**})) = i^{**}(y^{**})$. Note that

$$\begin{aligned}\pi^{**}(g)(i^{**}(y^{**})) &= \pi(g)^{**}(i^{**}(y^{**})) = y^{**} \circ i^* \circ \pi(g)^*,\\ i^{**}(y^{**}) &= y^{**} \circ i^*,\end{aligned}$$

so we need to prove that $y^{**} \circ i^* \circ \pi(g)^* = y^{**} \circ i^*$. Let $x^* \in X^*$. Observe that

$$\begin{aligned}(y^{**} \circ i^* \circ \pi(g)^*)(x^*) &= y^{**}(i^*(\pi(g)^*(x^*)))\\ &= y^{**}(\pi(g)^*(x^*)|_{X^G})\\ &= y^{**}((x^* \circ \pi(g))|_{X^G})\\ &= y^{**}(x^*|_{X^G})\\ &= y^{**}(i^*(x^*))\\ &= (y^{**} \circ i^*)(x^*).\end{aligned}$$

Next, we will show that $i^{**}\left(\left(X^G\right)^{**}\right) \supseteq (X^{**})^G$. Let $x^{**} \in (X^{**})^G$ and let $p : X \to X^G$ be any linear projection. Define $y^{**} := x^{**} \circ p^*$. It is not difficult to check that $y^{**} \in \left(X^G\right)^{**}$ and

$$i^{**}(y^{**}) = y^{**} \circ i^* = x^{**} \circ p^* \circ i^* = x^{**}.$$

(5) First off, $\overline{X^G}^{w^*} = \left(X^G\right)^{**}$. Now, i^{**} is w^*-w^* continuous and $i^{**}|_{X^G} = i$ (see Subsection 2.1), therefore according to items (2), (3), and (4) above we have that

$$(X^{**})^G \supseteq \overline{X^G}^{w^*} = \overline{i\left(X^G\right)}^{w^*} = \overline{i^{**}\left(X^G\right)}^{w^*} \supseteq i^{**}\left(\overline{X^G}^{w^*}\right) = i^{**}\left(\left(X^G\right)^{**}\right) = (X^{**})^G.$$

□

Proposition 3.2. *Let X be a vector space over a division ring K and consider a representation $\pi : G \to \mathsf{GL}(X)$ of the group G. Define $X_G := \left((X^*)^G\right)^\top = \left((X^*)^G\right)^\perp \cap X$. Then:*

1. *X_G is a G-invariant vector subspace of X.*
2. *If $K = \mathbb{R}$ or $\mathbb{C}$, then $(X_G)^\perp = (X^*)^G$.*
3. *If K is topological, then $(X^*)_G = \left(X^G\right)^\perp$.*
4. *For all $g \in G$ we have that $(\pi(g) - \pi(e))(X) \subseteq X_G$.*

Proof.

1. Let $x \in X_G$ and $g \in G$. Fix any arbtiray $x^* \in (X^*)^G$. Then
$$\begin{aligned} x^*(\pi(g)(x)) &= (x^* \circ \pi(g))(x) \\ &= \pi(g)^*(x^*)(x) \\ &= \pi^*\left(g^{-1}\right)(x^*)(x) \\ &= x^*(x) \\ &= 0, \end{aligned}$$
which means that $\pi(g)(x) \in \left((X^*)^G\right)^\perp \cap X = X_G$.
2. It suffices to notice that
$$(X_G)^\perp = \left(\left((X^*)^G\right)^\top\right)^\perp = \overline{(X^*)^G}^{w^*} = (X^*)^G$$
in virtue of (2) of Lemma 2.1 and (2) of Proposition 3.1.
3. Simply take into account (5) of Proposition 3.1 to conclude that
$$\left(X^G\right)^\perp = \left(\overline{X^G}^{w^*}\right)^\perp \cap X^* = \left((X^{**})^G\right)^\perp \cap X^* = (X^*)_G.$$
4. Let $x \in X$ and fix any arbtiray $x^* \in (X^*)^G$. Following the same reasoning as in the proof of (1) of this proposition we have that
$$x^*(\pi(g)(x) - x) = x^*(x) - x^*(x) = 0,$$
therefore $\pi(g)(x) - x \in \left((X^*)^G\right)^\perp \cap X = X_G$.

□

The next result, which will not be proved, has also an important impact on the further subsections. Sometimes, when A and B are non-empty sets, $\mathcal{F}(A, B)$ stands for the set of functions from A to B.

Corollary 3.3. *Let X be a vector space over a division ring K and consider a representation $\pi : G \to \mathsf{GL}(X)$ of the group G. Then:*

1. *For every $g \in G$, the map*
$$\begin{array}{rccl} \rho_g : & X & \to & X_G \\ & x & \mapsto & \rho_g(x) := \pi(g)(x) - x \end{array}$$
is a linear operator verifying that:

(a) $X^G \subseteq \ker(\rho_g)$.
(b) $\rho_{gh} = \rho_g \circ \pi(h) + \rho_h$ *for all* $g, h \in G$.
(c) $\rho_g \circ \rho_h = \pi(gh) - \pi(g) - \pi(h) + \pi(e)$.

Note also that $X^G = \bigcap_{g\in G} \ker(\rho_g)$.

2. *The map*

$$\begin{array}{rccl} \rho: & X & \to & \mathcal{F}(G, X_G) \\ & x & \mapsto & \begin{array}{rccl} \rho(x): & G & \mapsto & X_G \\ & g & \mapsto & \rho(x)(g) := \rho_g(x) \end{array} \end{array}$$

is a linear operator such that $\ker(\rho) = X^G$.

3.3. G-invariant vectors in isomorphic representations

By making use of the Hahn-Banach Separation Theorem we can somehow provide an expression of the subspace X_G.

Theorem 3.4. *Let* $\pi : G \to \mathrm{Iso}(X)$ *be an isomorphic representation of the group* G *on the Hausdorff locally convex real or complex topological vector space* X*. Then:*

1. X^G *and* X_G *are both closed in* X*.*
2. $\overline{\mathrm{span}}\{(\pi(g) - \pi(e))(X) : g \in G\} = X_G$.
3. $\overline{\mathrm{span}}\{(\pi(g) - \pi(e))(X) : g \in G\}^\perp = (X^*)^G$.

Proof. Only item (2) will be proved since (1) is obvious and (3) is an inmediate consequence of (2) of this theorem and (2) of Proposition 3.2. By applying (4) of Proposition 3.2 and (1) of this theorem we have that $\overline{\mathrm{span}}\{(\pi(g) - \pi(e))(X) : g \in G\} \subseteq X_G$. Now assume to the contrary that there exists

$$x \in X_G \setminus \overline{\mathrm{span}}\{(\pi(g) - \pi(e))(X) : g \in G\}.$$

In accordance to the Hahn-Banach Separation Theorem we can find $x^* \in X^*$ such that $\mathrm{Re}\, x^*(x) > \sup \mathrm{Re}\, x^*(\overline{\mathrm{span}}\{(\pi(g) - \pi(e))(X) : g \in G\})$, which means that $x^*((\pi(g) - \pi(e))(X)) = \{0\}$ for all $g \in G$. We will show now that $x^* \in (X^*)^G$, which will contradict the fact that $x \in X_G$ because $\mathrm{Re}\, x^*(x) > 0$. Indeed, if $g \in G$ and $y \in X$, then $\pi^*(g(x^*))(y) = x^*\left(\pi\left(g^{-1}\right)(y)\right) = x^*(y)$. This implies that $\pi^*(g)(x^*) = x^*$ and $x^* \in (X^*)^G$. □

The reader may observe that item (3) of the above theorem appears implicitly shown in the proof of (2).

Corollary 3.5. *Let* $\pi : G \to \mathrm{Iso}(X)$ *be an isomorphic representation of the group* G *on the Hausdorff locally convex real or complex topological vector space* X*. Assume that* G *is generated by a set* $S = S^{-1}$*. Then:*

1. $\mathrm{span}\{(\pi(g) - \pi(e))(X) : g \in G\} = \mathrm{span}\{(\pi(s) - \pi(e))(X) : s \in S\}$.
2. $X_G = \overline{\mathrm{span}}\{(\pi(s) - \pi(e))(X) : s \in S\}$.
3. *If* S *consists of reflections, that is,* $s^2 = e$ *for all* $s \in S$*, then* $\rho_s^2 = -2\rho_s$ *and thus* $P_s := \frac{-1}{2}\rho_s$ *are continuous linear projections for all* $s \in S$*.*

Proof.

1. Let $g \in G$ and $s_1, \ldots, s_n \in S$ such that $g = s_1 \cdots s_n$. If $x \in X$, then

$$\begin{array}{rcl} \pi(g)(x) - (x) & = & \pi(s_1 \cdots s_n)(x) - (x) \\ & = & \pi(s_1 \cdots s_n)(x) - \pi(s_2 \cdots s_n)(x) + \pi(s_2 \cdots s_n)(x) - (x) \\ & = & \cdots \\ & = & \displaystyle\sum_{k=1}^{n-2} \pi(s_k)(\pi(s_{k+1} \cdots s_n)(x)) - \pi(s_{k+1} \cdots s_n)(x) \\ & + & \pi(s_{n-1})(\pi(s_n)(x)) - \pi(s_n)(x) + \pi(s_n)(x) - x \\ & \in & \mathrm{span}\{(\pi(s) - \pi(e))(X) : s \in S\}. \end{array}$$

2. It is an inmediate consequence of the previous item and (2) of Theorem 3.4.
3. It directly follows from (1) of Corollary 3.3.

□

We will finalize this subsection with the following corollary, the details of whose proof we spare to the reader.

Corollary 3.6. *Let $\pi : G \to \mathrm{Iso}(X)$ be an isomorphic representation of the group G on the Hausdorff locally convex real or complex topological vector space X. Then:*

1. *For every $g \in G$, ρ_g is continuous.*
2. *If G is a topological group and π is continuous when $\mathrm{Iso}(X)$ is endowed with the pointwise convergence topology, then $\rho(x)$ is continuous for all $x \in X$.*

3.4. G-invariant vectors in isometric representations

Inevitably a large a-mount of Banach Space Theory techniques will be applied here in order to accomplish the main results in this section. We will start off with the following technical lemma, which relates the G-invariant vectors of the predual with the G-invariant vectors of the dual. We refer the reader to Subsection 2.3 for a proper understanding of the geometrical concepts involved in the upcoming result.

Lemma 3.7. *Let $\pi : G \to \mathcal{G}_X$ be an isometric representation of the group G on the real or complex normed space X. Consider an element $x \in \mathsf{S}_X$ and the w^*-exposed face of B_{X^*} supported by x, that is, $\mathsf{C}_x := x^{-1}(1) \cap \mathsf{B}_{X^*}$. Then:*

1. *$\pi^*(g)(\mathsf{C}_x) = \mathsf{C}_{\pi(g)(x)}$ for all $g \in G$.*
2. *If $x \in X^G$, then C_x is a G-invariant w^*-exposed face of B_{X^*}.*
3. *If $x \in \mathrm{smo}(\mathsf{B}_X) \cap X^G$, then $\mathsf{J}_X(x) \in (X^*)^G$.*

Proof. The reader may inmediately notice that (2) is a direct consequence of (1) and so is (3) of (2). Therefore we only ought to show (1), which is nothing else but a straight application of Equation (2.1). Indeed, if $g \in G$, then

$$\pi^*(g)(\mathsf{C}_x) = \pi\left(g^{-1}\right)^*(\mathsf{C}_x) = \mathsf{C}_{\left(\left(\pi(g^{-1})^*\right)^{-1}\right)^*(x)} = \mathsf{C}_{\pi(g^{-1})^{-1}(x)} = \mathsf{C}_{\pi(g)(x)}.$$

□

What follows is one of the main results on this section and finds part of its foundations on Theorem 2.7.

Theorem 3.8. *Let $\pi : G \to \mathcal{G}_X$ be an isometric representation of the group G on the real or complex normed space X. Then:*

1. *If $\mathsf{S}_{X^G} \subseteq \mathrm{smo}(\mathsf{B}_X)$, then $\|x\| \le \|x + y\|$ for all $x \in X^G$ and all $y \in X_G$.*
2. *If $\mathsf{S}_{(X^*)^G} \subseteq \mathrm{smo}(\mathsf{B}_{X^*})$, then $(X^*)^G + (X^*)_G$ is closed in X^*.*
3. *If $\mathsf{S}_{(X^*)^G} \subseteq \mathrm{smo}(\mathsf{B}_{X^*})$, then $X^G + X_G$ is dense in X.*

Proof.

1. Let $x \in X^G$ and $y \in X_G$. In accordance with Lemma 3.7 we have that $\mathsf{J}_X(x) \in (X^*)^G$ and thus $\mathsf{J}_X(x)(y) = 0$. As a consequence

$$\begin{aligned}\|x\|^2 &= \mathsf{J}_X(x)(x)\\ &= \mathsf{J}_X(x)(x+y)\\ &\le \|\mathsf{J}_X(x)\|\,\|x+y\|\\ &= \|x\|\,\|x+y\|.\end{aligned}$$

2. First off, both $(X^*)^G$ and $(X^*)_G$ are w^*-closed in X^* in virtue of (2) of Proposition 3.1 and (3) of Proposition 3.2, respectively. Now it suffices to apply (1) of this theorem together with Theorem 2.7.
3. In virtue of (1) of this theorem, (1) of Theorem 2.7, and (2) and (3) of Proposition 3.2, we have that
$$\left(X^G + X_G\right)^{\perp} = \left(X^G\right)^{\perp} \cap (X_G)^{\perp} = (X^*)_G \cap (X^*)^G = \{0\}\,.$$
Finally apply (1) of Lemma 2.1 to conclude that $X^G + X_G$ is dense in X.

□

The following corollary strongly relies on Theorem 2.7.

Corollary 3.9. *Let $\pi : G \to \mathcal{G}_X$ be an isometric representation of the group G on the real or complex normed space X. Then:*

- *If $\mathsf{S}_{X^G} \subseteq \operatorname{smo}(\mathsf{B}_X)$, then X^G and X_G are 1-complemented in $X^G + X_G$.*
- *If $\mathsf{S}_{X^G} \subseteq \operatorname{smo}(\mathsf{B}_X)$ and $\mathsf{S}_{(X^*)^G} \subseteq \operatorname{smo}(\mathsf{B}_{X^*})$, then X^G and X_G are quasi-complemented in X.*
- *If $\mathsf{S}_{(X^{**})^G} \subseteq \operatorname{smo}(\mathsf{B}_{X^{**}})$ and $\mathsf{S}_{(X^*)^G} \subseteq \operatorname{smo}(\mathsf{B}_{X^*})$, then $(X^*)^G$ and $(X^*)_G$ are 1-complemented in X^*.*

Corollary 3.10. *Let $\pi : G \to \mathcal{G}_X$ be an isometric representation of the group G on the real or complex normed space X. Then:*

1. *For every $g \in G$, $\|\rho_g\| \leq 2$.*
2. *For every $x \in X$, $\rho(x) \in \ell_\infty(G, X_G)$ and $\|\rho(x)\|_\infty \leq 2\|x\|$.*
3. *$\rho : X \to \ell_\infty(G, X_G)$ is a continuous linear operator with $\|\rho\| \leq 2$ and $\ker(\rho) = X^G$.*

The reader may notice that in oder to accompish (3) of Corollary 3.10 it is not necessary that G be a topological group or π be continuous.

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