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Some weighted integral inequalities for differentiable h-preinvex functions

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Abstract

In this paper, we present weighted integral inequalities of Hermite-Hadamard type for differentiable h-preinvex functions. We have established the weighted generalization of recent results for preinvex functions as well as we extend several results connected with the Hermite-Hadamard type integral inequalities by weighted identity of functions defined on open invex subset of set of reals and by using h-preinvexity.

Keywords: Hermite-Hadamard's inequality, invex set, preinvex function, h-preinvex function.

1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex in the classical sense, if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)y$$

holds for every $x, y \in I$ and $t \in [0, 1]$.
Indeed, a vast literature has been written on inequalities using classical convexity but one of the most celebrated is the Hermite-Hadamard inequality.
This double inequality is stated as follows:
Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $a, b \in I$ with $a < b$. Then f is convex on $[a, b]$ if and only if

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

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holds (see [18]). Both the inequalities in (1.1) hold in reversed direction if f is concave. Inequalities (1.1) are famous in mathematical literature due to their rich geometrical significance and applications.

For several results which generalize, improve and extend inequalities (1.1), we refer the interested reader to [1, 5, 6, 8, 9, 10, 11, 13, 14, 17, 19, 20, 21, 22, 23, 24, 25, 28].

In [6], Dragomir and Agarwal obtained the following inequalities for differentiable functions which estimate the difference between the middle and the rightmost terms in (1.1).

Theorem 1.1 ([6]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°. If $\left|f'\right|$ is convex function on $[a,b]$, with a, $b \in I$ and $a < b$, and $f' \in L([a,b])$ then following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{8}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \tag{1.2}$$

Theorem 1.2 ([6]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°. If $\left|f'\right|^{\frac{p}{p-1}}$ is a convex function on $[a,b]$, , with a, $b \in I$ and $a < b$,and $f' \in L([a,b])$ the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}}\right], \tag{1.3}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

In [17], Pearce and Pečarić gave an refined and simplified form of the constant in Theorem 1.2 and these results are strengthen with Theorem 1.1. The following is the main result from [17].

Theorem 1.3 ([17]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°. If $\left|f'\right|^q$is a convex function on $[a,b]$, for some $q \ge 1$, with a, $b \in I$ and $a < b$, and $f' \in L([a,b])$ the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}. \tag{1.4}$$

If $\left|f'\right|^q$ is concave on $[a,b]$, for some $q \ge 1$. Then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{1.5}$$

2. Inequalities for h-preinvex functions

A significant generalization of convex functions termed preinvex functions was introduced by Weir, Mond and Jeyakumar. Hanson has introduced a new class of generalized convex functions, subsequently it was called by Craven as invex functions, with the aim to extend the validity of the sufficiency of the Kuhn-Tucker conditions.

Since the papers of Hanson and Craven, many authors have studied invex functions, and their generalizations and related functions.

Here we are presenting some basic definitions of preinvex and h-preinvex functions before we proceed towards our main results. [7, 27]

Definition 2.1 ([4]). Let K be a subset in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}^n$ be continuous functions. Let $a \in K$, then the set K is said to be invex at a with respect to $\eta(\cdot,\cdot)$, if

$$a + t\eta(b,a) \in K, \forall a, b \in K, t \in [0,1].$$

K is said to be an invex set with respect to η if K is invex at each $a \in K$. The invex set K is also called a η-connected set.

This definition essentially says that there is a path starting from a point a which is contained in K. The point b may not be one of the end points of the path. This observation plays an important role in our analysis. If b is an end point of the path for every pair of points $a, b \in K$, then $\eta(b,a) = b - a$ and consequently, invexity reduces to convexity.
Thus, it is true that every convex set is also an invex set with respect to $\eta(b,a) = b - a$, but not conversely (see Mohan and Neogy, (1995), Weir and Mond, (1988) and related references therein).
For the sake of simplicity, we always assume that $K = [a, a + \eta(b,a)]$, unless otherwise specified.

Definition 2.2 ([26]). The function f on the invex set K is said to be preinvex with respect to η, if

$$f(a + t\eta(b,a)) \le (1-t) f(a) + t f(b), \forall a, b \in K, t \in [0,1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta(b,a) = b - a$ but the converse is not true see for instance [28].

Theorem 2.3 ([15]). *Let $f : [a, a+\eta(b,a)] \to (0,\infty)$ be a preinvex function on the interval of the real numbers K° (the interior of K) and a, $b \in K^\circ$ with $\eta(b,a) > 0$. Then the following inequalities holds:*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \le \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{2.1}$$

Theorem 2.4 ([3]). *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $\left|f'\right|$ is preinvex on K, for every $a, b \in K$ with $\eta(b,a) \neq 0$, the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx\right| \le \frac{|\eta(b,a)|}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right). \tag{2.2}$$

Theorem 2.5 ([3]). *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. Assume $p \in \mathbb{R}$ with $p > 1$. If $\left|f'\right|^{\frac{p}{p-1}}$ is preinvex on K, for every $a, b \in K$ with $\eta(b,a) \neq 0$, the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx\right| \le \frac{|\eta(b,a)|}{2(1+p)^{\frac{1}{p}}}\left[\frac{\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{2.3}$$

Theorem 2.6. *Let $f : [a, a+\eta(b,a)] \to (0,\infty)$ be a preinvex function on the interval of the real numbers K° (the interior of K) and a, $b \in K^\circ$ with $\eta(b,a) > 0$. Then the following inequalities holds:*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \le \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{2.4}$$

The double inequality Equation (2.4) is known as the Hermite-Hadamard-Noor inequality for prepinvex functions [16].
If $\eta(b,a) = b - a$, then the double inequality Equation (2.4) reduces to the classical Hermite-Hadamard inequality for convex functions.

Definition 2.7 ([12]). The function $f : K \to [0,\infty)$ on the invex set $K \subseteq [0,\infty)^n$ is said to be s-preinvex with respect to η, if

$$f(a + t\eta(b,a)) \le (1-t)^s f(a) + t^s f(b), \forall a, b \in K, t \in [0,1].$$

for some fixed $s \in (0,1]$.
The function f is said to be s- preincave if and only if $-f$ is preinvex.

Definition 2.8 ([15])**.** Let $h : [0, 1] \to \mathbb{R}$ be a nonnegative function, $h \neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η, if

$$f(a + \eta(b, a)) \leq h(1 - t)f(a) + h(t)f(b)$$

for each $a, b \in K$ and $t \in [0, 1]$ where $f(.) > 0$. If the above inequality is reversed, then f is said to be h-preconcave. Note, that every convex function is a h-preinvex function with respect to $\eta(b, a) = b - a$ and $h(t) = t$ for any $t \in [0, 1]$.

Definition 2.9 ([12])**.** A function $h : K \to \mathbb{R}$ is said to be a super-additive function if

$$h(a + b) \geq h(a) + h(b)$$

for all $a, b \in K$, when $a + b \in K$

Theorem 2.10 ([15])**.** *Let $f : K = [a, a + \eta(b, a)] \to \mathbb{R}$ be a differential function on I°, such that $f' \in L^1([a, a + \eta(b, a)])$, where $a < a + \eta(b, a)$. If $|f'|$ is h-preinvex on $[a, a + \eta(b, a)]$, then we have*

$$\left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \\ \leq \frac{\eta(b, a)}{2} \left[\left| f'(a) \right| + \left| f'(b) \right| \right] \int_{\frac{1}{2}}^1 (2t - 1)(h(t) + h(1 - t))\, dt. \quad (2.5)$$

Theorem 2.11 ([12])**.** *Let $f : K = [a, a + \eta(b, a)] \to \mathbb{R}$ be a differential function on I°, such that $f' \in L^1([a, a + \eta(b, a)])$, where $a < a + \eta(b, a)$. If $|f'|^q$ is h-preinvex on $[a, a + \eta(b, a)]$, and $q \geq 1$, then we have*

$$\left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \\ \leq \frac{\eta(b, a)}{2} \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\left| f'(a) \right|^q + \left| f'(b) \right|^q \right)^{\frac{1}{q}} \left(\int_0^1 (2t - 1)(h(t) + h(1 - t))\, dt \right)^{\frac{1}{q}}. \quad (2.6)$$

For several recent results on inequalities for preinvex and h-preinvex functions, we refer the interested readers to [12, 15, 16, 26, 29].

3. Main results

The following Lemma is essential in establishing our main results in this section:

Lemma 3.1 ([12])**.** *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and a, $b \in K$ with $\eta(b, a) > 0$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a + \eta(b, a)])$. If $h : [a, a + \eta(b, a)] \to [0, \infty)$ be a differentiable mapping, then the following equality holds:*

$$\frac{1}{2} \left[(h(a + \eta(b, a)) - 2h(a)) f(a) + h(a + \eta(b, a)) f(a + \eta(b, a)) \right] - \int_a^{a+\eta(b,a)} f(x) h'(x)\, dx = \frac{\eta(b, a)}{4} \\ \times \left\{ \int_0^1 \left[2h\left(a + \left(\frac{1 - t}{2}\right)\eta(b, a)\right) - h(a + \eta(b, a)) \right] f'\left(a + \left(\frac{1 - t}{2}\right)\eta(b, a)\right) dt \right. \\ \left. + \int_0^1 \left[2h\left(a + \left(\frac{1 + t}{2}\right)\eta(b, a)\right) - h(a + \eta(b, a)) \right] f'\left(a + \left(\frac{1 + t}{2}\right)\eta(b, a)\right) dt \right\}. \quad (3.1)$$

Remark 3.2. If we take $\eta(b, a) = b - a$, then Lemma 3.1 reduces to Lemma 2.1 from [8].

Now using Lemma 3.1, we shall propose some new upper bounds for the difference between the rightmost and middle terms of weighted version of the Hadamard's inequality (2.4) using pre-invex and pre-quasiinvex mappings. Our results provide a weighted generalization of those results given in [2, 3] and [29].

In what follows we use the notations $L^{'}(a,b,t)=a+\left(\frac{1-t}{2}\right)\eta(b,a)$ and $U^{'}(a,b,t)=a+\left(\frac{1+t}{2}\right)\eta(b,a)$.

Theorem 3.3. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$ and $a,\ b\in K$ with $\eta(b,a)>0$. Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K and $w:[a,a+\eta(b,a)]\to[0,\infty)$ be continuous and symmetric to $a+\frac{1}{2}\eta(b,a)$. If $\left|f^{'}\right|$ is h-preinvex on K, we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)}w(x)\,dx-\int_a^{a+\eta(b,a)}f(x)w(x)\,dx\right|$$
$$\leq\frac{\eta(b,a)}{4}\left[\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right]\int_0^1\left(\int_{L^{'}(a,b,t)}^{U^{'}(a,b,t)}w(x)\,dx\right)\left[h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right]dt. \quad (3.2)$$

Proof. Let $h(t)=\int_a^t w(t)\,dt$ for all $t\in[a,a+\eta(b,a)]$ in Lemma 3.1, we obtain

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)}w(x)\,dx-\int_a^{a+\eta(b,a)}f(x)w(x)\,dx\right|$$
$$\leq\frac{\eta(b,a)}{4}\left\{\int_0^1\left|2h\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)-h(a+\eta(b,a))\right|\right.$$
$$\times\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|dt$$
$$\left.+\int_0^1\left|2h\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)-h(a+\eta(b,a))\right|\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|dt\right\}. \quad (3.3)$$

Since $w(x)$ is symmetric to $a+\frac{1}{2}\eta(b,a)$, so

$$w\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)=w\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)$$

and hence, we have

$$\left|2h\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)-h(a+\eta(b,a))\right|=\int_{L^{'}(a,b,t)}^{U^{'}(a,b,t)}w(x)\,dx \quad (3.4)$$

and

$$\left|2h\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)-h(a+\eta(b,a))\right|=\int_{L^{'}(a,b,t)}^{U^{'}(a,b,t)}w(x)\,dx \quad (3.5)$$

for all $t\in[0,1]$. Using (3.4) and (3.5) in (3.3), we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)}w(x)\,dx-\int_a^{a+\eta(b,a)}f(x)w(x)\,dx\right|$$
$$\leq\frac{\eta(b,a)}{4}\int_0^1\left(\int_{L^{'}(a,b,t)}^{U^{'}(a,b,t)}w(x)\,dx\right)\left[\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|\right.$$
$$\left.+\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|\right]dt. \quad (3.6)$$

Since $\left|f'\right|$ is h-preinvex on K, hence for every $a, b \in K$ with $\eta(b,a) > 0$, we have

$$\begin{aligned}
&\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right| + \left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right| \\
&\quad\leq h\left(\frac{1+t}{2}\right)\left|f'(a)\right| + h\left(\frac{1-t}{2}\right)\left|f'(b)\right| + h\left(\frac{1-t}{2}\right)\left|f'(a)\right| + h\left(\frac{1+t}{2}\right)\left|f'(b)\right| \\
&\quad= \left(\left|f'(a)\right| + \left|f'(b)\right|\right)\left(h\left(\frac{1+t}{2}\right) + h\left(\frac{1-t}{2}\right)\right). \qquad (3.7)
\end{aligned}$$

Using (3.7) in (3.6), we get the required inequality. This completes the proof of the theorem. □

Corollary 3.4. *In Theorem 3.3, if we take* $w(x) = \frac{1}{\eta(b,a)}$ *for all* $x \in [a, a+\eta(b,a)]$, *then* (3.2) *becomes the inequality:*

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \\
&\quad\leq \frac{\eta(b,a)}{4}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]\int_0^1 t\left[h\left(\frac{1+t}{2}\right) + h\left(\frac{1-t}{2}\right)\right]dt. \qquad (3.8)
\end{aligned}$$

Corollary 3.5. *If* $\eta(b,a) = b-a$ *in Theorem 3.3, then* (3.2) *reduces to the inequality:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}\int_0^1 w(x)dx - \int_a^b f(x)\,w(x)\,dx\right| \\
&\quad\leq \frac{(b-a)}{4}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]\int_0^1 \left(\int_{L(a,b,t)}^{U(a,b,t)} w(x)dx\right)\left[h\left(\frac{1+t}{2}\right) + h\left(\frac{1-t}{2}\right)\right]dt. \qquad (3.9)
\end{aligned}$$

where $U(a,b,t) = \frac{1-t}{2}a + \frac{1+t}{2}b$ *and* $L(a,b,t) = \frac{1+t}{2}a + \frac{1-t}{2}b$ *for all* $t \in [0,1]$

Corollary 3.6. *If* $\eta(b,a) = b-a$, $w(x) = \frac{1}{b-a}$ *in Theorem 3.3, then* (3.2) *reduces to the inequality:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{4}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]\int_0^1 t\left[h\left(\frac{1+t}{2}\right) + h\left(\frac{1-t}{2}\right)\right]dt. \qquad (3.10)$$

Corollary 3.7. *Suppose* $h(t) = t^s$, $s \in [0,1]$ *in Corollary 3.4, we have the following inequality for s-convex function:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{\eta(b,a)\left(s2^{s+1}+1\right)}{2^{s+2}(s+1)(s+2)}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \qquad (3.11)$$

Corollary 3.8. *If* $\eta(b,a) = b-a$ *in Corollary 3.7, we have the following inequality for s-convex function:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{(b-a)}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)\left(s2^{s+1}+1\right)}{2^{s+2}(s+1)(s+2)}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \qquad (3.12)$$

Corollary 3.9. *Suppose the assumptions of Theorem 3.3 are satisfied. If h is super additive, we have the following inequality:*

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx - \int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right| \\
&\quad\leq \frac{\eta(b,a)\,h(1)}{4}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]\int_0^1 \left(\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right)dt. \qquad (3.13)
\end{aligned}$$

Theorem 3.10. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a,\ b \in K$ with $\eta(b,a) > 0$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K and $w : [a, a+\eta(b,a)] \to [0,\infty)$ be continuous and symmetric to $a + \frac{1}{2}\eta(b,a)$. If $\left|f'\right|^q$ is h-preinvex on K for $q > 1$, we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx - \int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right| \le \frac{\eta(b,a)}{2}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}$$
$$\times\left[\int_0^1\left(h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right)dt\right]^{\frac{1}{q}}\left(\int_0^1\left[\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right]^p dt\right)^{\frac{1}{p}}, \quad (3.14)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Continuing from inequality (3.6) in the proof of Theorem 3.3 and using the well known Hölder's integral inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx - \int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right|$$
$$\le \frac{\eta(b,a)}{4}\left(\int_0^1\left[\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right]^p dt\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right]. \quad (3.15)$$

By the power-mean inequality $t^r + s^r < 2^{1-r}(t+s)^r$ for $t > 0$, $s > 0$ and $r < 1$, and by the h-preinvexity of $\left|f'\right|^q$ on K for $q > 1$, we have for every $a, b \in K$ with $\eta(b,a) > 0$ the following inequality

$$\left(\int_0^1\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\le 2^{1-\frac{1}{q}}\left[\int_0^1\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt+\int_0^1\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right]^{\frac{1}{q}}$$
$$\le 2^{1-\frac{1}{q}}\left[\int_0^1\left\{h\left(\frac{1+t}{2}\right)\left|f'(a)\right|^q+h\left(\frac{1-t}{2}\right)\left|f'(b)\right|^q\right.\right.$$
$$\left.\left.+h\left(\frac{1-t}{2}\right)\left|f'(a)\right|^q+h\left(\frac{1+t}{2}\right)\left|f'(b)\right|^q\right\}dt\right]^{\frac{1}{q}}$$
$$=2^{1-\frac{1}{q}}\left[\int_0^1\left(h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right)dt\right]^{\frac{1}{q}}\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]^{\frac{1}{q}}. \quad (3.16)$$

Using the last inequality (3.16) in (3.15), we get the desired inequality. This completes the proof of the theorem as well. □

Corollary 3.11. *In Theorem 3.10 if we take $w(x) = \frac{1}{\eta(b,a)}$ for all $x \in [a, a+\eta(b,a)]$ with $\eta(b,a) > 0$, then* (3.14) *reduces to the inequality*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2(1+p)^{\frac{1}{p}}}\left[\frac{\left|f'(a)\right|^{\frac{p}{p-1}}+\left|f'(b)\right|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}\left(\int_0^1\left[h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right]dt\right)^{\frac{p-1}{p}}. \quad (3.17)$$

Corollary 3.12. *If we take $\eta(b,a)=b-a$ in Theorem 3.10, then (3.14) reduces to the following inequality:*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(x)\,dx-\int_a^b f(x)\,w(x)\,dx\right|\leq\frac{b-a}{2}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}$$
$$\times\left(\int_0^1\left[h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right]dt\right)^{\frac{1}{q}}\left(\int_0^1\left[\int_{L(a,b,t)}^{U(a,b,t)}w(x)\,dx\right]^p\right)^{\frac{1}{p}}dt, \quad (3.18)$$

where $\frac{1}{p}+\frac{1}{q}=1$, $L(a,b,t)=\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b$, $U(a,b,t)=\left(\frac{1-t}{2}\right)a+\left(\frac{1+t}{2}\right)b$, $t\in[a,b]$.

Corollary 3.13. *Assume that all the conditions of Theorem 3.10 are satisfied and in addition if h is super-additive, we have the following inequality*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)}w(x)\,dx-\int_a^{a+\eta(b,a)}f(x)\,w(x)\,dx\right|$$
$$\leq\frac{\eta(b,a)\,(h(1))^{\frac{1}{q}}}{2}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\left(\int_0^1\left[\int_{L'(a,b,t)}^{U'(a,b,t)}w(x)\,dx\right]^p dt\right)^{\frac{1}{p}}. \quad (3.19)$$

Corollary 3.14. *Suppose $h(t)=t^s$, $s\in[0,1]$ in Corollary 3.11, we have the following inequality for s-convex function.*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|$$
$$\leq\frac{\eta(b,a)}{2^{1-\frac{s}{q}}(1+p)^{\frac{1}{p}}(1+s)^{\frac{1}{q}}}\left[\frac{\left|f'(a)\right|^{\frac{p}{p-1}}+\left|f'(b)\right|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \quad (3.20)$$

A similar result may be stated as follows:

Theorem 3.15. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$. Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K and $w:[a,a+\eta(b,a)]\to[0,\infty)$ be continuous and symmetric to $a+\frac{1}{2}\eta(b,a)$. If $\left|f'\right|^q$ is h-preinvex on K for $q\geq 1$, then for every $a,b\in K$ with $\eta(b,a)>0$, we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)}w(x)\,dx-\int_a^{a+\eta(b,a)}f(x)\,w(x)\,dx\right|$$
$$\leq\frac{\eta(b,a)}{2}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\left(\int_0^1\int_{L'(a,b,t)}^{U'(a,b,t)}w(x)\,dxdt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left(\int_{L'(a,b,t)}^{U'(a,b,t)}w(x)\,dx\right)\left[h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right]dt\right)^{\frac{1}{q}}. \quad (3.21)$$

Proof. Continuing from inequality (3.6) in the proof of Theorem 3.3 and using the well known Hölder's integral inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx-\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right| \\
&\leq \frac{\eta(b,a)}{4}\left[\int_0^1\left(\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right)dt\right]^{1-\frac{1}{q}} \\
&\times\left[\left(\int_0^1\left(\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right)\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left(\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right)\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right]. \quad (3.22)
\end{aligned}$$

By the power-mean inequality $t^r+s^r<2^{1-r}(t+s)^r$ for $t>0$, $s>0$ and $r<1$, and by the h-preinvexity of $\left|f'\right|^q$ on K for $q>1$, we have for every $a,b\in K$ with $\eta(b,a)>0$ the following inequality

$$\begin{aligned}
&\left(\int_0^1\left(\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right)\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\left(\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right)\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq 2^{1-\frac{1}{q}}\left[\int_0^1\left(\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dx\right)dt\right]^{\frac{1}{q}}\left[h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right]^{\frac{1}{q}} \\
&\times\left[\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]^{\frac{1}{q}}. \quad (3.23)
\end{aligned}$$

Utilizing inequality (3.23) in (3.22), we get the inequality (3.29). This completes the proof of the theorem. □

Corollary 3.16. *Suppose all the assumptions of Theorem 3.15 are satisfied and if* $w(x)=\frac{1}{\eta(b,a)}$ *for all* $x\in[a,a+\eta(b,a)]$ *with* $\eta(b,a)>0$*, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \\
&\leq \frac{\eta(b,a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\left(\int_0^1 t\left[h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right]dt\right)^{\frac{1}{q}}. \quad (3.24)
\end{aligned}$$

Corollary 3.17. *If we take* $\eta(b,a)=b-a$ *and* $w(x)=\frac{1}{(b-a)}$ *in Theorem 3.15, then the inequality reduces to the inequality:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)\,dx\right| \\
&\leq \frac{(b-a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\left[\int_0^1 t\left(h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right)dt\right]^{\frac{1}{q}}. \quad (3.25)
\end{aligned}$$

Corollary 3.18. *In Corollary 3.16, put $h(t) = t$, then we have:*

$$\left|\frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \le \frac{\eta(b,a)}{4}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}. \quad (3.26)$$

Corollary 3.19. *Under the assumptions of Theorem 3.15, if h is super-additive, then we have the following inequality:*

$$\left|\frac{f(a) + f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx - \int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right|$$
$$\le \frac{\eta(b,a)\,(h(1))^{\frac{1}{q}}}{2}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}\left(\int_0^1\int_{L'(a,b,t)}^{U'(a,b,t)} w(x)\,dxdt\right). \quad (3.27)$$

Corollary 3.20. *If h is super-additive in Corollary 3.16, then we have the following inequality:*

$$\left|\frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \le \frac{\eta(b,a)\,h(1)^{\frac{1}{q}}}{4}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}. \quad (3.28)$$

Corollary 3.21. *If h is super-additive in Corollary 3.17, then we have the following inequality:*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{(b-a)}\int_a^{b} f(x)\,dx\right| \le \frac{(b-a)h(1)^{\frac{1}{q}}}{4}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}. \quad (3.29)$$

Corollary 3.22. *Suppose $h(t) = t^s$, $s \in [0,1]$ in Corollary 3.16, we have the following inequality for s-convex function:*

$$\left|\frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\le \frac{\eta(b,a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{\left(s2^{s+1}+1\right)}{2^s\,(s+1)\,(s+2)}\right)^{\frac{1}{q}}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}. \quad (3.30)$$

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