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Existence of global solutions via invariant regions for a generalized reaction-diffusion system with a tri-diagonal Toeplitz matrix of diffusion coefficients

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Abstract

The aim of this paper is to construct invariant regions of a generalized m-component reaction-diffusion system with a tri-diagonal Toeplitz matrix of diffusion coefficients and prove the global existence of solutions using Lyapunov functional. The paper assumes nonhomogeneous boundary conditions and polynomial growth for the non-linear reaction term.

Keywords: Reaction-diffusion systems, invariant regions, global existence.
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1. Introduction

Reaction-diffusion systems arise in many applications ranging from chemistry and biology to engineering. They have been the subject of countless studies in the past few decades. One of the most important aspects of this broad field is proving the global existence of solutions under certain assumptions and restrictions. We quote the recent papers of Amann [4, 5] who studies the problem in $W^{1,p}$ and $W^{2,p}$ spaces with $p > n$. An excellent reference for a dynamic theory of reaction-diffusion systems is the book of Henry [11].

In 2001, Kouachi [13] followed on previous work and showed the global existence of solutions assuming the reaction terms of a 2×2 diagonal system exhibit a polynomial growth. This was later generalized by Kouachi for an arbitrary 2×2 Toeplitz matrix. In [1], the author of this work studied the 3×3 case under the same assumptions and restrictions. Abdelmalek and Kouachi [3] also showed the global existence

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of solutions for an m-component reaction–diffusion system ($m \geq 2$) with a diagonal diffusion matrix and reaction terms of polynomial growth.

An important factor in the study of reaction diffusion systems is the characteristics of the diffusion matrix. Although in some cases the matrix is diagonal, in many cases cross diffusion terms exist. For instance, many chemical and biological operations are described by reaction-diffusion systems with a tri-diagonal matrix of diffusion coefficients, (see, e.g., Cussler [8] and [9]). Other examples include the modelling of epidemics [7], ecology [16] and biochemistry [6], where cross-diffusion appears to be a very relevant problem to be analyzed. In this paper, tri-diagonal diffusion matrices have been considered and sufficient conditions have been given for global existence steady states.

The purpose of this paper is to prove the global existence of solutions with nonhomogeneous Neumann, Dirichlet, or Robin conditions and a polynomial growth of reaction terms. The polynomial growth is established through a mere single inequality as we shall show. The main contribution of this paper is the fact that we take a general Toeplitz matrix as opposed to the symmetry constraint assumed in [2].

Throughout this paper, we consider an m-component system, with $m \geq 2$:

$$\frac{\partial U}{\partial t} - D\Delta U = F(U) \text{ in } \Omega \times (0, +\infty), \tag{1.1}$$

with the boundary conditions:

$$\alpha U + (1-\alpha)\,\partial_\eta U = B \text{ on } \partial\Omega \times (0, +\infty), \tag{1.2}$$

or

$$\alpha U + (1-\alpha)\,D\partial_\eta U = B \text{ on } \partial\Omega \times (0, +\infty), \tag{1.3}$$

in the case of non-diagonal boundary conditions, and the initial data:

$$U(x, 0) = U_0(x) \text{ on } \Omega. \tag{1.4}$$

We consider three types of boundary conditions:

(i) Nonhomogeneous Robin boundary conditions, corresponding to

$$0 < \alpha < 1,\ B \in \mathbb{R}^m;$$

(ii) Homogeneous Neumann boundary conditions, corresponding to

$$\alpha = 0 \text{ and } B \equiv 0;$$

(iii) Homogeneous Dirichlet boundary conditions, corresponding to

$$1 - \alpha = 0 \text{ and } B \equiv 0.$$

In the context of this work, Ω is an open bounded domain of class C^1 in $\mathbb{R}^n$ with boundary $\partial\Omega$, $\frac{\partial}{\partial\eta}$ denotes the outward normal derivative on $\partial\Omega$, and

$$U := (u_1, \ldots, u_m)^T,$$
$$F := (f_1, \ldots, f_m)^T,$$
$$B := (\beta_1, \ldots, \beta_m)^T.$$

The diffusion matrix is assumed to be a tri-diagonal Toeplitz one of the form

$$D := \begin{pmatrix} a & b & 0 & \cdots & 0 \\ c & a & b & \ddots & \vdots \\ 0 & c & a & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b \\ 0 & \cdots & 0 & c & a \end{pmatrix}_{m\times m},$$

where a, b and c are supposed to be strictly positive constants satisfying:

$$\cos\frac{\pi}{m+1} < \frac{a}{b+c}, \tag{1.5}$$

which reflects the parabolicity of the system.

The initial data are assumed to be in the regions:

$$\Sigma_{\mathfrak{L},\mathfrak{Z}} := \{U_0 \in \mathbb{R}^m : \langle V_z, U_0\rangle \leq 0 \leq \langle V_\ell, U_0\rangle\,,\ \ell \in \mathfrak{L},\ z \in \mathfrak{Z}\}\,, \tag{1.6}$$

$$\mathfrak{L} \cap \mathfrak{Z} = \varnothing, \mathfrak{L} \cup \mathfrak{Z} = \{1, 2, \ldots, m\}\,, \tag{1.7}$$

subject to

$$\langle V_z, B\rangle \leq 0 \leq \langle V_\ell, B\rangle\,,\ \ell \in \mathfrak{L}, z \in \mathfrak{Z}.$$

The vector $V_\ell = (v_{1\ell}, \ldots, v_{m\ell})^T$ are defined as

$$v_{k\ell} = \sqrt{\mu^k}\sin\frac{k\,(m+1-\ell)\,\pi}{m+1},\ k = 1, \ldots, m,$$

with

$$\mu := \frac{b}{c}.$$

The notation $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^m$.

From (1.7) we can clearly see that there are in fact 2^m regions. One of the main contributions of this paper is that unlike previous studies we cover *all* possible regions. Hence, the work carried out here is a generalization of previous studies. The most important of these studies are discussed below.

In 2002, Kouachi [14] studied the case $m = 2$, for which the parabolicity condition we use here (1.5) reduces to the same condition employed in [14]: $2a > (b + c)$. Although in this case $2^2 = 4$ regions exist, the study of Kouachi considered only a couple of these regions. Setting $m = 2$ in (1.6) yields the following regions:

- If $\mathfrak{L} = \{1, 2\}\,, \mathfrak{Z} = \emptyset$ then,

$$\Sigma_{\mathfrak{L},\mathfrak{Z}} = \left\{\left(u_1^0, u_2^0\right)^T \in \mathbb{R}^2 : u_1^0 \geq \sqrt{\mu}\left|u_2^0\right| \text{ if } \beta_1 \geq \sqrt{\mu}\,|\beta_2|\right\}.$$

- If $\mathfrak{L} = \{2\}\,, \mathfrak{Z} = \{1\}$ then,

$$\Sigma_{\mathfrak{L},\mathfrak{Z}} = \left\{\left(u_1^0, u_2^0\right)^T \in \mathbb{R}^2 : \sqrt{\mu}u_2^0 \geq \left|u_1^0\right| \text{ if } \sqrt{\mu}\beta_2 \geq |\beta_1|\right\}.$$

- If $\mathfrak{L} = \emptyset, \mathfrak{Z} = \{1, 2\}$ then,

$$\Sigma_{\mathfrak{L},\mathfrak{Z}} = \left\{\left(u_1^0, u_2^0\right)^T \in \mathbb{R}^2 : -u_1^0 \geq \sqrt{\mu}\left|u_2^0\right| \text{ if } -\beta_1 \geq \sqrt{\mu}\,|\beta_2|\right\}.$$

- If $\mathfrak{L} = \{1\}\,, \mathfrak{Z} = \{2\}$ then,

$$\Sigma_{\mathfrak{L},\mathfrak{Z}} = \left\{\left(u_1^0, u_2^0\right)^T \in \mathbb{R}^2 : -\sqrt{\mu}u_2^0 \geq \left|u_1^0\right| \text{ if } -\sqrt{\mu}\beta_2 \geq |\beta_1|\right\}.$$

In fact the last two of these regions were not considered in [14].

In 2007, the author of this work [1] studied the case $m = 3$ for which the parabolicity condition is $\sqrt{2}a > (b + c)$, resulting from the direct substitution of $m = 3$ in (1.5). The total number of regions in this case is $2^3 = 8$ of which only 4 regions were however studied.

In 2014 the author [2] elaborated on the generalized m-component case with a tri-diagonal matrix having equal upper and lower diagonal elements, i.e. $(b = c)$. Substituting $b = c$ in (1.5) yields the same condition used in [2]: $2b \cos \frac{\pi}{m+1} < a$.

The aim of this work is to prove the global existence of solutions. The necessary proofs are similar for all the invariant regions. Hence we only focus on one of the regions and present a generalization at the end of the paper.

Consider the region with $\mathfrak{L} = \{1, 2, \ldots, m\}$ and $\mathfrak{Z} = \emptyset$ yielding

$$\Sigma_{\mathfrak{L},\emptyset} = \{U_0 \in \mathbb{R}^m : \langle V_\ell, U_0 \rangle \geq 0, \ \ell \in \mathfrak{L}\}, \tag{1.8}$$

subject to

$$\langle V_\ell, B \rangle \geq 0, \ \ell \in \mathfrak{L}.$$

In order to establish the global existence of solutions in this region we diagonalize the diffusion matrix D. We define the reaction diffusion functions as:

$$\digamma(W) := (\digamma_1, \digamma_2, \ldots, \digamma_m)^T, \ \digamma_\ell := \langle V_\ell, F \rangle, \tag{1.9}$$

where the variable $W = (w_1, w_2, \ldots, w_m)^T$ is given by

$$W := (w_1, w_2, \ldots, w_m)^T, w_\ell := \langle V_\ell, U \rangle. \tag{1.10}$$

The functions $\digamma_\ell$ must satisfy the following three conditions:

(A1) be continuously differentiable on $\mathbb{R}_+^m$ for all $\ell = 1, \ldots, m$, satisfying $\digamma_\ell(w_1, \ldots, w_{\ell-1}, 0, w_{\ell+1}, \ldots, w_m) \geq 0$, for all $w_\ell \geq 0$; $\ell = 1, \ldots, m$.

(A2) be of polynomial growth (see the work of Hollis and Morgan [12]), which means that for all $\ell = 1, \ldots, m$:

$$|\digamma_\ell(W)| \leq C_1 (1 + \langle W, 1 \rangle)^N, n \in \mathbb{N}, \text{on } (0, +\infty)^m. \tag{1.11}$$

(A3) satisfy the inequality:

$$\langle S, \digamma(W) \rangle \leq C_2 (1 + \langle W, 1 \rangle), \tag{1.12}$$

where

$$S := (d_1, d_2, \ldots, d_{n-1}, 1)^T,$$

for all $w_\ell \geq 0$, $\ell = 1, \ldots, m$. All the constants d_ℓ satisfy $d_\ell \geq \overline{d_\ell}$, $\ell = 1, \ldots, m$ where $\overline{d_\ell}$, $\ell = 1, \ldots, m$, are sufficiently large positive constants. Here C_1 and C_2 are uniformly bounded positive functions defined on $\mathbb{R}_+^m$.

The following sections of this paper are organized as follows: Section 2 presents some important propositions and lemmas regarding properties of the diffusion matrix and the parabolicity of the system. Section 3 identifies one invariant region for the proposed system and establishes the local existence of solutions through the diagonalization of the proposed system. Section 4 establishes the global existence of solutions for the equivalent diagonalized system through the use of an appropriate Lyapunov functional. The last section identifies the remaining invariant regions and refers to the trivial generalization of the work carried out here to all the regions.

2. Some properties of the diffusion matrix and parabolicity

Before we look at the diagonalization of our system and establish the existence of solutions locally and globally, it is important to state some important properties that will aid our proofs later on.

Proposition 2.1. *A quadratic form $Q = \langle X, AX \rangle = X^T AX$, with A being a symmetric matrix, is positive definite for every non-zero column vector X if all the principal minors in the top-left corner of A are positive. If A is non-symmetric, Q is positive definite iff the principal minors in the top-left corner of $\frac{1}{2}(A + A^T)$ are positive.*

Lemma 2.2. *The reaction-diffusion system* (1.1) *satisfies the parabolicity condition if* (1.5) *is satisfied.*

Proof. The system (1.1) satisfies the parabolicity condition if the matrix $\left(D+D^T\right)$ is positive definite. The matrix $\left(D+D^T\right)$ is symmetric tri-diagonal with off-diagonal elements $\frac{1}{2}(b+c)$. In [2] a similar matrix with off-diagonal elements b and the parabolicity condition

$$2b\cos\frac{\pi}{m+1}<a,$$

is considered. Substituting b with $\frac{1}{2}(b+c)$ yields (1.5). □

Lemma 2.3 ([15])**.** *The eigenvalues $\overline{\lambda}_\ell<\overline{\lambda}_{\ell-1}$; $\ell=2,\dots,m,$ of D^T are positive and are given by*

$$\overline{\lambda}_\ell := a+2\sqrt{bc}\cos\left(\frac{\ell\pi}{m+1}\right), \tag{2.1}$$

with the corresponding eigenvectors being $\overline{V}_\ell=V_{m+1-\ell}$, for $\ell=1,\dots,m$. Therefore, D^T is diagonalizable.

In the remainder of this work we require an ascending order of the eigenvalues. In order to simplify the indices in the formulas to come we define

$$\lambda_\ell := \bar{\lambda}_{m+1-\ell}=a+2\sqrt{bc}\cos\left(\frac{(m+1-\ell)\pi}{m+1}\right);\ \ell=1,\dots,m, \tag{2.2}$$

thus $\lambda_\ell<\lambda_{\ell+1}$; $\ell=2,\dots,m.$

Proof. Recall that the diffusion matrix is positive definite, implying that its eigenvalues are necessarily positive. For a given eigenpair $\left(\overline{\lambda},X\right)$ the components of $\left(D^T-\overline{\lambda}I\right)X=0$ are

$$bx_{k-1}+\left(a-\overline{\lambda}\right)x_k+cx_{k+1}=0,\ k=1,\dots,m,$$

with $x_0=x_{m+1}=0$, or equivalently,

$$x_{k+2}+\left(\frac{a-\overline{\lambda}}{c}\right)x_{k+1}+\mu x_k=0, k=0,\dots,m-1,$$

whose solutions are

$$x_k=\begin{cases}\alpha r_1^k+\beta r_2^k, & \text{if } r_1\neq r_2,\\ \alpha\rho^k+\beta k\rho^k, & \text{if } r_1=r_2=\rho,\end{cases}$$

where α and β are arbitrary constants. For the eigenvalue problem at hand, r_1 and r_2 must be distinct. Putting $x_k=\alpha r_1^k+\beta r_2^k$, and $x_0=x_{m+1}=0$ yields

$$\begin{cases}0=\alpha+\beta\\ 0=\alpha r_1^{m+1}+\beta r_2^{m+1}\end{cases}\Rightarrow\left(\frac{r_1}{r_2}\right)^{m+1}=\frac{-\beta}{\alpha}=1\Rightarrow\frac{r_1}{r_2}=e^{\frac{2i\pi\ell}{m+1}}.$$

Therefore we see that $r_1=r_2e^{\frac{2i\pi\ell}{m+1}}$ for $1\le\ell\le m$. This together with

$$r^2+\left(\frac{a-\overline{\lambda}}{c}\right)r+\mu=(r-r_1)(r-r_2)\Rightarrow\begin{cases}r_1r_2=\mu\\ r_1+r_2=-\frac{a-\overline{\lambda}}{c}\end{cases},$$

leads to $r_1=\sqrt{\mu}e^{\frac{i\pi\ell}{m+1}}$, $r_2=\sqrt{\mu}e^{-\frac{i\pi\ell}{m+1}}$, and

$$\overline{\lambda}=a+2\sqrt{cb}\left(e^{\frac{i\pi\ell}{m+1}}+e^{-\frac{i\pi\ell}{m+1}}\right)=a+2a+2\sqrt{cb}\cos\left(\frac{\ell\pi}{m+1}\right).$$

Thus the eigenvalues of D^T are given by

$$\overline{\lambda}_\ell = a + 2\sqrt{cb}\cos\left(\frac{\ell\pi}{m+1}\right),$$

for $\ell = 1, \ldots, m$. Since the eigenvalues are all distinct (as $\cos\theta$ is strictly decreasing on $(0,\pi)$, and $b \neq 0 \neq c$), then D is necessarily diagonalizable. The ℓ^{th} component of any eigenvector associated with λ_ℓ satisfies $x_k = \alpha r_1^k + \beta r_2^k$, with $\alpha + \beta = 0$. Thus

$$x_k = \alpha\mu^{\frac{k}{2}}\left(e^{\frac{2i\pi k}{m+1}} - e^{-\frac{2i\pi k}{m+1}}\right) = 2i\alpha\mu^{\frac{k}{2}}\sin\left(\frac{k}{m+1}\pi\right).$$

Setting $\alpha = \frac{1}{2i}$ yields a particular eigenvector associated to $\overline{\lambda}_\ell$ given by

$$\overline{V}_\ell = \left(\mu^{\frac{1}{2}}\sin\left(\frac{1\ell\pi}{m+1}\right), \mu^{\frac{2}{2}}\sin\left(\frac{2\ell\pi}{m+1}\right), \ldots, \mu^{\frac{m}{2}}\sin\left(\frac{m\ell\pi}{m+1}\right)\right)^t.$$

Since the eigenvectors are all distinct then $\{\overline{V}_1, \overline{V}_2, \ldots, \overline{V}_m\}$ is a complete linearly independent set, hence $\left(\overline{V}_1 \mid \overline{V}_2 \mid \ldots \mid \overline{V}_m\right)$ *diagonalizes* D. Now let us prove that

$$\overline{\lambda}_\ell < \overline{\lambda}_{\ell-1};\ \ell = 2, \ldots, m.$$

We have

$$\ell > \ell - 1 \Rightarrow \frac{\ell\pi}{m+1} > \frac{(\ell-1)\pi}{m+1}.$$

Once again using the fact that $\cos\theta$ is strictly decreasing on $(0,\pi)$, we deduce that

$$\cos\left(\frac{\ell\pi}{m+1}\right) < \cos\left(\frac{(\ell-1)\pi}{m+1}\right),$$

whereupon

$$\overline{\lambda}_\ell = a + 2\sqrt{cb}\cos\left(\frac{\ell\pi}{m+1}\right) < a + 2\sqrt{cb}\cos\left(\frac{(\ell-1)\pi}{m+1}\right) = \overline{\lambda}_{\ell-1}.$$

□

Lemma 2.4. *The eigenvalues of the matrix D are positive, i.e. $\lambda_\ell > 0$ and $\det D > 0$.*

Proof. Recall that $\lambda_\ell < \lambda_{\ell+1}$; $\ell = 1, \ldots, m-1$, i.e.

$$\lambda_1 < \lambda_2 < \ldots < \lambda_m.$$

We want to show that $\lambda_1 > 0$. First, we have

$$\lambda_1 = a + 2\sqrt{cb}\cos\left(\frac{m}{m+1}\pi\right) > 0, \tag{2.3}$$

which implies

$$a > 2\sqrt{bc}\left[-\cos\left(\frac{m}{m+1}\pi\right)\right].$$

From condition (1.5), we obtain

$$a > (c+b)\left(\cos\frac{\pi}{m+1}\right). \tag{2.4}$$

Note that

$$\frac{m}{m+1} > \frac{1}{2} \Rightarrow \cos\left(\frac{m}{m+1}\pi\right) < \cos\frac{\pi}{2} = 0; \tag{2.5}$$

furthermore

$$(c+b) \geq 2\sqrt{bc}. \tag{2.6}$$

We also have

$$\cos\left(\frac{\pi}{m+1}\right) + \cos\left(\frac{m}{m+1}\pi\right) = 0 \tag{2.7}$$

since

$$\begin{aligned}\cos\left(\frac{\pi}{m+1}\right) + \cos\left(\frac{m}{m+1}\pi\right) &= 2\cos\left(\frac{\frac{\pi}{m+1} + \frac{m}{m+1}\pi}{2}\right)\cos\left(\frac{\frac{\pi}{m+1} - \frac{m}{m+1}\pi}{2}\right)\\ &= 2\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{m-1}{m+1}\frac{\pi}{2}\right) = 0.\end{aligned}$$

Now, from (2.5), (2.6), and (2.7), we obtain

$$(c+b)\left(\cos\frac{\pi}{m+1}\right) \geq 2\sqrt{bc}\left[-\cos\left(\frac{m}{m+1}\pi\right)\right], \tag{2.8}$$

and from (2.4) and (2.8), we get

$$a > 2\sqrt{bc}\left[-\cos\left(\frac{m}{m+1}\pi\right)\right],$$

which concludes the proof of (2.3) and guarantees that all eigenvalues of D^T are positive. Furthermore since the eigenvalues of D are the same as those of D^T we conclude that $\det D > 0$. □

3. Local existence and invariant regions

In this section, we use the eigenvalues and eigenvectors of the diffusion matrix to diagonalize the proposed system and establish the local existence of solutions. First, the usual norms in spaces $L^p(\Omega)$, $L^\infty(\Omega)$ and $C(\overline{\Omega})$ are denoted respectively by:

$$\begin{aligned}\|u\|_p^p &= \frac{1}{|\Omega|}\int_\Omega |u(x)|^p\, dx;\\ \|u\|_\infty &= \underset{x\in\Omega}{ess\sup}\, |u(x)|,\end{aligned}$$

and

$$\|u\|_{C(\overline{\Omega})} = \max_{x\in\overline{\Omega}} |u(x)|.$$

It is well-known that in order to prove the global existence of solutions to a reaction-diffusion system (see Henry [11]) it suffices to derive a uniform estimate of the associated reaction term on $[0, T_{\max})$ in the space $L^p(\Omega)$ for some $p > n/2$. Our aim is to construct Lyapunov polynomial functionals allowing us to obtain L^p-bounds on the components, which leads to global existence. Since the reaction terms are continuously differentiable on $\mathbb{R}_+^m$, then for any initial data in $C(\overline{\Omega})$ it is straightforward to directly check their Lipschitz continuity on bounded subsets of the domain of a fractional power of the operator

$$\mathfrak{D} = -\operatorname{diag}(\lambda_1\Delta, \lambda_2\Delta, \ldots, \lambda_m\Delta). \tag{3.1}$$

The assumption (1.5) implies that $D\Delta$ is a strongly elliptic operator in the sense of Petrowski, see Friedman [10].

Proposition 3.1. *Diagonalizing system* (1.1) *yields:*

$$W_t - diag(\lambda_1, \lambda_2, \ldots, \lambda_m)\Delta W = F(W) \;\; in\; \Omega\times(0,+\infty), \tag{3.2}$$

with the boundary condition

$$\alpha W + (1-\alpha)\,\partial_\eta W = \Lambda \qquad on\ \partial\Omega \times (0,+\infty)\,, \tag{3.3}$$

or

$$\alpha W + (1-\alpha)\,diag\,(\lambda_1,\lambda_2,\ldots,\lambda_m)\,\partial_\eta W = \Lambda\ on\ \partial\Omega \times (0,+\infty)\,, \tag{3.4}$$

and the initial data

$$W\,(x,0) = W_0 \qquad on\ \Omega$$

Proof. The eigenvectors of the diffusion matrix associated with the eigenvalues λ_ℓ are defined as $V_\ell = (v_{\ell 1}, v_{\ell 2},\ldots,v_{\ell m})^T$. Let us consider the diagonalizing matrix of eigenvectors $P = (V_1 \mid V_2 \mid \ldots \mid V_m)$ and define the solution vector U and the reaction terms vector F. Pre-multiplying the system by P^T yields

$$\begin{aligned} U_t - D\Delta U &= F \\ P^T U_t - \Delta P^T D U &= P^T F \\ P^T U_t - \Delta P^T D \left(P^T\right)^{-1} P^T U &= P^T F. \end{aligned} \tag{3.5}$$

The term $P^T U$ can be simplified as follows

$$\begin{aligned} P^T U &= (V_1 \mid V_2 \mid \ldots \mid V_m)^T\, U \\ &= (\langle V_1, U\rangle\,, \langle V_2, U\rangle\,, \ldots, \langle V_m, U\rangle)^T \\ &= (w_1, w_2,\ldots,w_m)^T = W. \end{aligned} \tag{3.6}$$

Hence, $P^T U_t = W_t$. Similarly,

$$\begin{aligned} P^T F &= (V_1 \mid V_2 \mid \ldots \mid V_m)^T\, F \\ &= (\langle V_1, F\rangle\,, \langle V_2, F\rangle\,, \ldots, \langle V_m, F\rangle)^T \\ &= (\digamma_1, \digamma_2,\ldots,\digamma_m)^T = \digamma. \end{aligned} \tag{3.7}$$

Furthermore we have the similarity transformation

$$\begin{aligned} P^T D \left(P^T\right)^{-1} &= P^T \left(D^T\right)^T \left(P^{-1}\right)^T \\ &= \left(D^T P\right)^T \left(P^{-1}\right)^T \\ &= \left(P^{-1} D^T P\right)^T \\ &= (diag\,(\lambda_1,\lambda_2,\ldots,\lambda_m))^T \\ &= diag\,(\lambda_1,\lambda_2,\ldots,\lambda_m)\,. \end{aligned} \tag{3.8}$$

Substituting (3.6), (3.7), and (3.8) in (3.5) results in the equivalent system (3.2). The boundary condition (3.3) can be obtained by pre-multiplying (1.2) by P^T:

$$\begin{aligned} \alpha U + (1-\alpha)\,\partial_\eta U &= B, \\ \alpha P^T U + (1-\alpha)\,\partial_\eta P^T U &= P^T B. \end{aligned} \tag{3.9}$$

Simplifying the term $P^T B$ yields

$$\begin{aligned} P^T B &= (V_1 \mid V_2 \mid \ldots \mid V_m)^T\, B \\ &= (\langle V_1, B\rangle\,, \langle V_2, B\rangle\,, \ldots, \langle V_m, B\rangle)^T \\ &= \left(\rho_1^0, \rho_2^0,\ldots,\rho_m^0\right)^T := \Lambda. \end{aligned} \tag{3.10}$$

Substituting (3.6) and (3.9) in (3.10) gives the boundary condition for the equivalent system (3.3). Premultiplying (1.3) by P^T yields

$$\alpha U + (1-\alpha)\, D\partial_\eta U = B,$$
$$\alpha P^T U + (1-\alpha)\, \partial_\eta P^T D U = P^T B,$$
$$\alpha P^T U + (1-\alpha)\, \partial_\eta P^T D \left(P^T\right)^{-1} P^T U = P^T B. \tag{3.11}$$

Substituting (3.7), (3.8), and (3.10) results in the equivalent boundary condition in (3.4). We note that condition (1.5) guarantees the parabolicity of the system (1.1), which implies that this system is equivalent to that described by (3.2) in the region:

$$\Sigma_{\mathfrak{L},\mathfrak{Z}} = \{U_0 \in \mathbb{R}^m : \langle V_\ell, U_0\rangle \geq 0,\ \ell \in \mathfrak{L}\}$$
$$= \left\{U_0 \in \mathbb{R}^m : w_\ell^0 = \langle V_\ell, U_0\rangle \geq 0,\ \ell \in \mathfrak{L}\right\}$$

with

$$\rho_\ell^0 = \langle V_\ell, B\rangle \geq 0, \ell \in \mathfrak{L}.$$

This implies that the components w_ℓ are necessarily positive. □

The local existence and uniqueness of solutions to the initial system (1.1), with initial data in $C(\overline{\Omega})$ or $L^p(\Omega)$, $p \in (1, +\infty)$, follows from the basic existence theory for abstract semi-linear differential equations (Henry [11]). The solutions are classical on $(0, T_{\max})$, where $T_{\max}$ denotes the eventual blow up time in $L^\infty(\Omega)$. The local solution is continued globally by apriori estimates. Once the invariant regions are constructed, one can apply the Lyapunov technique and establish the global existence of a unique solution for (1.1).

Proposition 3.2. *The system* (3.2) *admits a unique classical solution* W *on* $\Omega \times (0, T_{\max})$*; moreover we have the alternative*

$$\text{If } T_{\max} < \infty \text{ then } \lim_{t \nearrow T_{\max}} \sum_{\ell=1}^{m} \left\| w_\ell (t, .) \right\|_\infty = \infty, \tag{3.12}$$

where $T_{\max}\left(\left\|w_1^0\right\|_\infty, \left\|w_2^0\right\|_\infty, \ldots, \left\|w_m^0\right\|_\infty\right)$ *denotes the eventual blow-up time.*

4. Main result

This section presents the main findings of this study. The aim is to show that subject to the stated conditions, solutions to the proposed system exist globally in time. This is done through the use of a Lyapunov functional. First, let us define

$$K_l^r = K_{r-1}^{r-1} K_l^{r-1} - \left[H_l^{r-1}\right]^2,\ r = 3, \ldots, l, \tag{4.1}$$

where

$$H_l^r = \det_{1\leq \ell,\kappa \leq l}\left(\left(a_{\ell,\kappa}\right)_{\substack{\ell \neq l,\ldots,r+1\\ \kappa \neq l-1,\ldots,r}}\right) \prod_{k=1}^{k=r-2} \left(\det\left[k\right]\right)^{2^{(r-k-2)}},\ r = 3, \ldots, l-1,$$

$$K_l^2 = \underbrace{\bar{\lambda}_1 \bar{\lambda}_l \prod_{k=1}^{l-1} \theta_k^{2(p_k+1)^2} \prod_{k=l}^{m-1} \theta_k^{2(p_k+2)^2}}_{\text{positive value}} \left[\prod_{k=1}^{l-1} \theta_k^2 - A_{1l}^2\right],$$

and

$$H_l^2 = \underbrace{\bar{\lambda}_1 \sqrt{\bar{\lambda}_2 \bar{\lambda}_l}\, \theta_1^{2(p_1+1)^2} \prod_{k=2}^{l-1} \theta_k^{(p_k+2)^2+(p_k+1)^2} \prod_{k=l}^{m-1} \theta_k^{2(p_k+2)^2}}_{\text{positive value}} \left[\theta_1^2 A_{2l} - A_{12} A_{1l}\right].$$

Here $\det\limits_{1\leq\ell,\kappa\leq l}\left((a_{\ell,\kappa})_{\substack{\ell\neq l,\ldots,r+1\\ \kappa\neq l-1,\ldots,r}}\right)$ denotes the determinant of the r-square symmetric matrix obtained from $(a_{\ell,\kappa})_{1\leq\ell,\kappa\leq m}$ by removing the $(r+1)^{\text{th}}$, $(r+2)^{\text{th}}$,…, l^{th} rows and the $r^{\text{th}}, (r+1)^{\text{th}}, \ldots, (l-1)^{\text{th}}$ columns, and $\det[1], \ldots, \det[m]$ are the minors of the matrix $(a_{\ell,\kappa})_{1\leq\ell,\kappa\leq m}$. The elements of the matrix are:

$$a_{\ell\kappa} = \frac{\lambda_\ell+\lambda_\kappa}{2}\theta_1^{p_1^2}\ldots\theta_{(\ell-1)}^{p_{(\ell-1)}^2}\theta_\ell^{(p_\ell+1)^2}\ldots\theta_{\kappa-1}^{\left(p_{(\kappa-1)}+1\right)^2}\theta_\kappa^{(p_\kappa+2)^2}\ldots\theta_{(m-1)}^{\left(p_{(m-1)}+2\right)^2}, \tag{4.2}$$

where λ_ℓ is defined in (2.1)–(2.2). Note that $A_{\ell\kappa} = \dfrac{\lambda_\ell+\lambda_\kappa}{2\sqrt{\lambda_\ell\lambda_\kappa}}$ for all $\ell, \kappa = 1, \ldots, m$, and θ_ℓ, $\ell = 1, \ldots, (m-1)$ are positive constants.

Theorem 4.1. *Suppose that the functions F_ℓ, $\ell = 1, \ldots, m$, are of polynomial growth and satisfy the condition* (1.12) *for some sufficiently large positive constants D_ℓ, $\ell = 1, \ldots, m$. Let $(w_1(t,.), w_2(t,.), \ldots, w_m(t,.))$ be a solution of* (3.2)–(3.3) *and*

$$L(t) = \int_\Omega H_{p_m}\left(w_1(t,x), w_2(t,x), \ldots, w_m(t,x)\right)\mathrm{d}x, \tag{4.3}$$

where

$$H_{p_m}(w_1,\ldots,w_m) = \sum_{p_{m-1}=0}^{p_m}\ldots\sum_{p_1=0}^{p_2} C_{p_m}^{p_{m-1}}\ldots C_{p_2}^{p_1}\theta_1^{p_1^2}\ldots\theta_{(m-1)}^{p_{(m-1)}^2} w_1^{p_1}w_2^{p_2-p_1}\ldots w_m^{p_m-p_{m-1}},$$

with p_m a positive integer and $C_{p_\kappa}^{p_\ell} = \frac{p_\kappa!}{p_\ell!(p_\kappa-p_\ell)!}$.
Furthermore suppose that the following condition is satisfied

$$K_l^l > 0, \; l = 2, \ldots, m. \tag{4.4}$$

where K_l^l was defined in (4.1)*. Then it follows that the functional L is uniformly bounded on the interval $[0, T^*]$, $T^* < T_{\max}$.*

Corollary 4.2. *Under the assumptions of Theorem 4.1, all solutions of* (3.2)–(3.3) *with positive initial data in $L^\infty(\Omega)$ are in $L^\infty(0, T^*; L^p(\Omega))$, for some $p \geq 1$.*

Proposition 4.3. *Under the assumptions of theorem 4.1 and given that the condition* (1.5) *is satisfied, all solutions of* (3.2)–(3.3) *with positive initial data in $L^\infty(\Omega)$ are global for some $p > \dfrac{Nn}{2}$.*

For the proof of Theorem 4.1, we first need to define some preparatory Lemmas.

Lemma 4.4 ([3])**.** *Let H_{p_m} be the homogeneous polynomial defined in* (4.3)*, we have*

$$\begin{aligned}\partial_{w_1}H_{p_m} = p_m &\sum_{p_{m-1}=0}^{p_m-1}\ldots\sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}}{}_{p_2}^{p_1}\theta_1^{(p_1+1)^2}\ldots\theta_{(m-1)}^{\left(p_{(m-1)}+1\right)^2}\\ &w_1^{p_1}w_2^{p_2-p_1}w_3^{p_3-p_2}{}_m^{(p_m-1)-p_{m-1}}.\end{aligned} \tag{4.5}$$

$$\begin{aligned}\partial_{w_\ell}H_{p_m} = p_m &\sum_{p_{m-1}=0}^{p_m-1}\ldots\sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}}\ldots C_{p_2}^{p_1}\theta_1^{p_1^2}\ldots\theta_{\ell-1}^{p_{(\ell-1)}^2}\theta_\ell^{(p_\ell+1)2}\ldots\theta_{(m-1)}^{\left(p_{(m-1)}+1\right)^2}\\ &w_1^{p_1}w_2^{p_2-p_1}w_3^{p_3-p_2}\ldots w_m^{(p_m-1)-p_{m-1}}, \;\; \ell = 2, \ldots, m-1,\end{aligned} \tag{4.6}$$

$$\begin{aligned}\partial_{w_m}H_{p_m} = p_m &\sum_{p_{m-1}=0}^{p_m-1}\ldots\sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}}\ldots C_{p_3}^{p_2}C_{p_2}^{p_1}\theta_1^{p_1^2}\theta_2^{p_2^2}\ldots\theta_{(m-1)}^{p_{(m-1)}^2}\\ &w_1^{p_1}w_2^{p_2-p_1}w_3^{p_3-p_2}\ldots w_m^{(p_m-1)-p_{m-1}}.\end{aligned} \tag{4.7}$$

Lemma 4.5 ([3])**.** *We have*

$$\partial_{w_1^2} H_n = p_m (p_m - 1) \sum_{p_{m-1}=0}^{p_m-2} \cdots \sum_{p_2=0}^{p_3} \sum_{p_1=0}^{p_2} C_{p_m-2}^{p_{m-1}} \cdots C_{p_2}^{p_1}$$
$$\theta_1^{(p_1+2)^2} \cdots \theta_{(m-1)}^{\left(p_{(m-1)}+2\right)^2} w_1^{p_1} w_2^{p_2-p_1} \cdots w_m^{(p_m-2)-p_{m-1}}, \tag{4.8}$$

$$\partial_{w_\ell^2} H_n = p_m (p_m - 1) \sum_{p_{m-1}=0}^{p_m-2} \cdots \sum_{p_1=0}^{p_2} C_{p_m-2}^{p_{m-1}} \cdots C_{p_2}^{p_1}$$
$$\theta_1^{p_1^2} \theta_2^{p_2^2} \cdots \theta_{\ell-1}^{p_{\ell-1}^2} \theta_\ell^{(p_\ell+2)^2} \cdots \theta_{(m-1)}^{\left(p_{(m-1)}+2\right)^2}$$
$$w_1^{p_1} w_2^{p_2-p_1} \cdots w_m^{(p_m-2)-p_{m-1}}, \tag{4.9}$$

for all $\ell = 2, \ldots, m-1$, *and*

$$\partial_{w_\ell w_\kappa} H_n = p_m (p_m - 1) \sum_{p_{m-1}=0}^{p_m-2} \cdots \sum_{p_1=0}^{p_2} C_{p_m-2}^{p_{m-1}} \cdots C_{p_2}^{p_1}$$
$$\theta_1^{p_1^2} \cdots \theta_{\ell-1}^{p_{\ell-1}^2} \theta_\ell^{(p_\ell+1)^2} \cdots \theta_{\kappa-1}^{(p_{\kappa-1}+1)^2} \theta_\kappa^{(p_\kappa+2)^2} \cdots \theta_{(m-1)}^{\left(p_{(m-1)}+2\right)^2}$$
$$w_1^{p_1} w_2^{p_2-p_1} \cdots w_m^{(p_m-2)-p_{m-1}} \tag{4.10}$$

for all $1 \leq \ell < \kappa \leq m$,

$$\partial_{w_m^2} H_n = p_m (p_m - 1) \sum_{p_{m-1}=0}^{p_m-2} \cdots \sum_{p_1=0}^{p_2} C_{p_m-2}^{p_{m-1}} \cdots C_{p_2}^{p_1} \theta_1^{p_1^2} \cdots \theta_{(m-1)}^{p_{(m-1)}^2}$$
$$w_1^{p_1} w_2^{p_2-p_1} \cdots w_m^{(p_m-2)-p_{m-1}}. \tag{4.11}$$

Lemma 4.6 ([3])**.** *Let A be the m-square symmetric matrix defined by $A = (a_{\ell\kappa})_{1\leq \ell,\kappa\leq m}$. Then the following property holds:*

$$\begin{cases} K_m^m = \det[m] \prod_{k=1}^{k=m-2} (\det[k])^{2^{(m-k-2)}}, \quad m > 2, \\ K_2^2 = \det[2], \end{cases} \tag{4.12}$$

where

$$K_m^l = K_{l-1}^{l-1} K_m^{l-1} - \left(H_m^{l-1}\right)^2, l = 3, \ldots, m,$$
$$H_m^l = \det_{1\leq \ell,\kappa\leq m}\left((a_{\ell,\kappa})_{\substack{\ell\neq m,\ldots,l+1 \\ \kappa\neq m-1,\ldots,l}}\right) \prod_{k=1}^{k=l-2} (\det[k])^{2^{(l-k-2)}}, \ l = 3, \ldots, m-1,$$
$$K_m^2 = a_{11} a_{mm} - (a_{1m})^2, \ H_m^2 = a_{11} a_{2m} - a_{12} a_{1m}.$$

Proof of Theorem 4.1. We prove that $L(t)$ is uniformly bounded on the interval $[0, T^*], T^* < T_{\max}$. We have:

$$\begin{aligned} L'(t) &= \int_\Omega \partial_t H_{p_m} \mathrm{d}x = \int_\Omega \sum_{\ell=1}^m \partial_{w_\ell} H_{p_m} \frac{\partial w_\ell}{\partial t} \mathrm{d}x \\ &= \int_\Omega \sum_{\ell=1}^m \partial_{w_\ell} H_{p_m} (\lambda_\ell \Delta w_\ell + F_\ell) \mathrm{d}x \\ &= \int_\Omega \sum_{\ell=1}^m \lambda_\ell \partial_{w_\ell} H_{p_m} \Delta w_\ell \mathrm{d}x + \int_\Omega \sum_{\ell=1}^m \partial_{w_\ell} H_{p_m} F_\ell \mathrm{d}x = I + J, \end{aligned}$$

where

$$I=\int_{\Omega}\sum_{\ell=1}^{m}\lambda_{\ell}\partial_{w_{\ell}}H_{p_m}\Delta w_{\ell}\mathrm{d}x, \tag{4.13}$$

and

$$J=\int_{\Omega}\sum_{\ell=1}^{m}\partial_{w_{\ell}}H_{p_m}F_{\ell}\mathrm{d}x. \tag{4.14}$$

Using Green's formula we can divide I into two parts: I_1 and I_2, where

$$I_1=\int_{\partial\Omega}\sum_{\ell=1}^{m}\lambda_{\ell}\partial_{w_{\ell}}H_{p_m}\partial_{\eta}w_{\ell}\mathrm{d}x, \tag{4.15}$$

and

$$I_2=-\int_{\Omega}\left\langle T,\left(\left(\frac{\lambda_{\ell}+\lambda_{\kappa}}{2}\partial_{w_{\kappa}w_{\ell}}H_{p_m}\right)_{1\leq\ell,\kappa\leq m}\right)T\right\rangle\mathrm{d}x, \tag{4.16}$$

for $p_1=0,\dots,p_2$, $p_2=0,\dots,p_3\ \dots p_{m-1}=0,\dots,p_m-2$ and
$T=(\nabla w_1,\nabla w_2,\dots,\nabla w_m)^T$. Applying Lemmas 4.4 and 4.5 yields

$$\left(\tfrac{\lambda_{\ell}+\lambda_{\kappa}}{2}\partial_{w_{\kappa}w_{\ell}}H_{p_m}\right)_{1\leq\ell,\kappa\leq m}=p_m\left(p_m-1\right)\sum_{p_{m-1}=0}^{p_m-2}\dots\sum_{p_1=0}^{p_2}C_{p_m-2}^{p_{m-1}}\dots C_{p_2}^{p_1}\left(\left(a_{\ell\kappa}\right)_{1\leq\ell,\kappa\leq m}\right)w_1^{p_1}\dots w_m^{(p_m-2)-p_{m-1}}, \tag{4.17}$$

where $(a_{\ell\kappa})_{1\leq\ell,\kappa\leq m}$ is the matrix defined in (4.2). Now, in order to prove that I is bounded, we will show that there exists a positive constant C_4 independent of $t\in[0,T_{\max})$ such that

$$I_1\leq C_4 \text{ for all } t\in\left[0,T_{\max}\right), \tag{4.18}$$

and that

$$I_2\leq 0, \tag{4.19}$$

for several boundary conditions. First let us prove (4.18): (i) If $0<\alpha<1$, then using the boundary conditions (1.2) we get

$$I_1=\int_{\partial\Omega}\sum_{\ell=1}^{m}\lambda_{\ell}\partial_{w_{\ell}}H_{p_m}\left(\gamma_{\ell}-\sigma_{\ell}w_{\ell}\right)\mathrm{d}s,$$

where $\sigma_{\ell}=\dfrac{\alpha}{1-\alpha}$ and $\gamma_{\ell}=\dfrac{\beta_{\ell}}{1-\alpha}$, for $\ell=1,\dots,m$. For the second type of boundary condition (3.4), $\sigma_{\ell}=\dfrac{\alpha}{\lambda_{\ell}\left(1-\alpha\right)}$ and $\gamma_{\ell}=\dfrac{\beta_{\ell}}{\lambda_{\ell}\left(1-\alpha\right)}$. Since $H\left(W\right)=\sum\limits_{\ell=1}^{m}\lambda_{\ell}\partial_{w_{\ell}}H_{p_m}\left(\gamma_{\ell}-\sigma_{\ell}w_{\ell}\right)=P_{n-1}\left(W\right)-Q_n\left(W\right)$, where P_{n-1} and Q_n are polynomials with positive coefficients and respective degrees $n-1$ and n, and since the solution is positive it follows that

$$\limsup_{\sum\limits_{\ell=1}^{m}|w_{\ell}|\to+\infty}H\left(W\right)=-\infty, \tag{4.20}$$

which proves that H is uniformly bounded on $\mathbb{R}_+^m$ and consequently proves (4.18). (ii) If for all $\ell=1,\dots,m$: $\alpha=0$, then $I_1=0$ on $[0,T_{\max})$. (iii) The case of homogeneous Dirichlet conditions is trivial since in this case the positivity of the solution on $[0,T_{\max})\times\Omega$ implies $\partial_{\eta}w_{\ell}\leq 0,\forall\ell=1,\dots,m$ on $[0,T_{\max})\times\partial\Omega$. Consequently one obtains the same result in (4.18) with $C_4=0$. Hence the proof of (4.18) is complete. Now we move to the proof of (4.19). Consider the matrix $(a_{\ell\kappa})_{1\leq\ell,\kappa\leq m}$ which we defined in (4.2). The quadratic form (with respect to ∇w_{ℓ}, $\ell=1,\dots,m$) associated with the matrix $(a_{\ell\kappa})_{1\leq\ell,\kappa\leq m}$, with $p_1=0,\dots,p_2$, $p_2=0,\dots,p_3$ $\dots p_{m-1}=0,\dots,p_m-2$, is positive definite since its minors det $[1]$, det $[2]$,… det $[m]$ are all positive. Let us prove their positivity by induction. The first minor

$$\det[1] = \lambda_1 \theta_1^{(p_1+2)^2} \theta_2^{(p_2+2)^2} \dots \theta_{(m-1)}^{\left(p_{(m-1)}+2\right)^2} > 0$$

for $p_1 = 0, \dots, p_2$, $p_2 = 0, \dots, p_3 \dots p_{m-1} = 0, \dots, p_m - 2$. For the second minor $\det[2]$, and according to Lemma 4.6, we have:

$$\det[2] = K_2^2 = \lambda_1 \lambda_2 \theta_1^{2(p_1+1)^2} \prod_{k=2}^{m-1} \theta_k^{2(p_k+2)^2} \left[\theta_1^2 - A_{12}^2\right],$$

using (4.4) for $l = 2$ we get $\det[2] > 0$. Similarly for the third minor $\det[3]$, and again using Lemma 4.6, we have:

$$K_3^3 = \det[3] \det[1].$$

Since $\det[1] > 0$, we conclude that

$$sign(K_3^3) = sign(\det[3]).$$

Again, using (4.4) for $l = 3$ yields $\det[3] > 0$. To finish the proof let us suppose $\det[k] > 0$ for $k = 1, 2, \dots, l-1$ and show that $\det[l]$ is necessarily positive. We have

$$\det[k] > 0, k = 1, \dots, (l-1) \Rightarrow \prod_{k=1}^{k=l-2} (\det[k])^{2^{(l-k-2)}} > 0. \tag{4.21}$$

From Lemma 4.6 we obtain $K_l^l = \det[l] \prod_{k=1}^{k=l-2} (\det[k])^{2^{(l-k-2)}}$, and from (4.21) we get $sign(K_l^l) = sign(\det[l])$. Since $K_l^l > 0$ according to (4.4) then $\det[l] > 0$ and the proof of (4.19) is concluded. It then follows from (4.18) and (4.19) that I is finished. Now let us prove that J in (4.14) is bounded. Substituting the expressions of the partial derivatives given by Lemma 4.4 in the second integral of (4.14) yields

$$\begin{aligned}
J &= \int_\Omega \left[p_m \sum_{p_{m-1}=0}^{p_m-1} \dots \sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}} \dots C_{p_2}^{p_1} w_1^{p_1} w_2^{p_2-p_1} \dots w_m^{p_m-1-p_{m-1}} \right] \\
&\left(\prod_{\ell=1}^{m-1} \theta_\ell^{(p_\ell+1)^2} F_1 + \sum_{\kappa=2}^{m-1} \prod_{k=1}^{\kappa-1} \theta_k^{p_k^2} \prod_{\ell=\kappa}^{m-1} \theta_\ell^{(p_\ell+1)^2} F_\kappa + \prod_{\ell=1}^{m-1} \theta_\ell^{p_\ell^2} F_m \right) \mathrm{d}x \\
&= \int_\Omega \left[p_m \sum_{p_{m-1}=0}^{p_m-1} \dots \sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}} \dots C_{p_2}^{p_1} w_1^{p_1} w_2^{p_2-p_1} \dots w_m^{p_m-1-p_{m-1}} \right] \\
&\left(\prod_{\ell=1}^{m-1} \frac{\theta_\ell^{(p_\ell+1)^2}}{\theta_\ell^{p_\ell^2}} F_1 + \sum_{\kappa=2}^{m-1} \prod_{k=1}^{\kappa-1} \theta_k^{p_k^2} \prod_{\ell=\kappa}^{m-1} \frac{\theta_\ell^{(p_\ell+1)^2}}{\theta_\ell^{p_\ell^2}} F_\kappa + F_m \right) \prod_{\ell=1}^{m-1} \theta_\ell^{p_\ell^2} \mathrm{d}x \\
&= \int_\Omega \left[p_m \sum_{p_{m-1}=0}^{p_m-1} \dots \sum_{p_1=0}^{p_2} C_{p_m-1}^{p_{m-1}} \dots C_{p_2}^{p_1} w_1^{p_1} w_2^{p_2-p_1} \dots w_m^{p_m-1-p_{m-1}} \right] \\
&\left\langle \left(\prod_{\ell=1}^{m-1} \frac{\theta_\ell^{(p_\ell+1)^2}}{\theta_\ell^{p_\ell^2}}, \theta_1^{p_1^2} \prod_{\ell=2}^{m-1} \frac{\theta_\ell^{(p_\ell+1)^2}}{\theta_\ell^{p_\ell^2}}, \dots, \prod_{k=1}^{m-2} \theta_k^{p_k^2} \frac{\theta_{m-1}^{(p_{m-1}+1)^2}}{\theta_{m-1}^{p_{m-1}^2}}, 1 \right), F \right\rangle \prod_{\ell=1}^{m-1} \theta_\ell^{p_\ell^2} \mathrm{d}x.
\end{aligned}$$

Hence using the condition (1.12) we deduce that

$$J \le C_5 \int_\Omega \left[\sum_{p_{m-1}=0}^{p_m-1} \dots \sum_{p_1=0}^{p_2} C_{p_2}^{p_1} \dots C_{p_m-1}^{p_{m-1}} w_1^{p_1} w_2^{p_2-p_1} \dots w_m^{p_m-1-p_{m-1}} \left(1 + \langle W, 1 \rangle\right) \right] \mathrm{d}x.$$

To prove that the functional L is uniformly bounded on the interval $[0,T^*]$ we write

$$\sum_{p_{m-1}=0}^{p_m-1}\dots\sum_{p_1=0}^{p_2}C_{p_2}^{p_1}\dots C_{p_m-1}^{p_{m-1}}w_1^{p_1}w_2^{p_2-p_1}\dots w_m^{p_m-1-p_{m-1}}\left(1+\langle W,1\rangle\right)=R_{p_m}\left(W\right)+S_{p_m-1}\left(W\right),$$

where $R_{p_m}\left(W\right)$ and $S_{p_m-1}\left(W\right)$ are two homogeneous polynomials of degrees p_m and p_m-1, respectively. Since all the polynomials H_{p_m} and R_{p_m} are of degree p_m then there exists a positive constant C_6 such that

$$\int_\Omega R_{p_m}\left(W\right)\mathrm{d}x\le C_6\int_\Omega H_{p_m}\left(W\right)\mathrm{d}x. \tag{4.22}$$

Applying Hölder's inequality to the integral $\int_\Omega S_{p_m-1}\left(W\right)\mathrm{d}x$, one obtains

$$\int_\Omega S_{p_m-1}\left(W\right)\mathrm{d}x\le\left(meas\Omega\right)^{\frac{1}{p_m}}\left(\int_\Omega\left(S_{p_m-1}\left(W\right)\right)^{\frac{p_m}{p_m-1}}\mathrm{d}x\right)^{\frac{p_m-1}{p_m}}.$$

Using the fact that for all $w_1,w_2,\dots,w_{m-1}\ge0$ and $w_m>0$,

$$\frac{\left(S_{p_m-1}\left(W\right)\right)^{\frac{p_m}{p_m-1}}}{H_{p_m}\left(W\right)}=\frac{\left(S_{p_m-1}\left(x_1,x_2,\dots,x_{m-1},1\right)\right)^{\frac{p_m}{p_m-1}}}{H_{p_m}\left(x_1,x_2,\dots,x_{m-1},1\right)},$$

where we have $\forall\ell\in\{1,2,\dots,m-1\}:x_\ell=\frac{w_\ell}{w_{\ell+1}}$, and

$$\lim_{x_\ell\to+\infty}\frac{\left(S_{p_m-1}\left(x_1,x_2,\dots,x_{m-1},1\right)\right)^{\frac{p_m}{p_m-1}}}{H_{p_m}\left(x_1,x_2,\dots,x_{m-1},1\right)}<+\infty,$$

one asserts that there exists a positive constant C_7 such that

$$\frac{\left(S_{p_m-1}\left(W\right)\right)^{\frac{p_m}{p_m-1}}}{H_{p_m}\left(W\right)}\le C_7,\ \text{for all } w_1,w_2,\dots,w_m\ge0. \tag{4.23}$$

Hence the functional L satisfies the differential inequality

$$L'\left(t\right)\le C_6L\left(t\right)+C_8L^{\frac{p_m-1}{p_m}}\left(t\right),$$

which for $Z=L^{\frac{1}{p_m}}$ can be written as

$$p_mZ'\le C_6Z+C_8. \tag{4.24}$$

A simple integration gives the uniform bound of the functional L on the interval $[0,T^*]$. This ends the proof of the theorem. □

Proof of Corollary 4.2. It is an immediate consequence of Theorem 4.1 and the inequality

$$\int_\Omega\langle W,1\rangle^p\,\mathrm{d}x\le C_9L\left(t\right)\ \text{on}\ \left[0,T^*\right]. \tag{4.25}$$

for some $p\ge1$. □

Proof of Proposition 4.3. From Corollary 4.2, it follows that there exists a positive constant C_{10} such that

$$\int_\Omega\left(\langle W,1\rangle+1\right)^p\mathrm{d}x\le C_{10}\ \text{on}\ \left[0,T_{\max}\right). \tag{4.26}$$

From (1.11), we have

$$\begin{gathered}\text{for any } \ell\in\{1,2,\dots,m\}:\\ \left|F_\ell\left(W\right)\right|^{\frac{p}{N}}\le C_{11}\left(W\right)\langle W,1\rangle^p\ \text{on}\ \left[0,T_{\max}\right)\times\Omega.\end{gathered} \tag{4.27}$$

Since $w_1,w_2,\dots,w_m$ are in $L^\infty\left(0,T^*;L^p\left(\Omega\right)\right)$ and $\dfrac{p}{N}>\dfrac{n}{2}$, then as discussed in section 2, the solution is global. □

5. Construction of invariant regions

The aim of this section is to identify all the existing invariant regions for the proposed system. Recall that the eigenvector of the diffusion matrix associated with the eigenvalue λ_ℓ is defined as $V_\ell = (v_{\ell 1}, v_{\ell 2}, \ldots, v_{\ell m})^T$. In the region that we considered in previous sections, we used the diagonalizing matrix $P = (V_1 \shortmid V_2 \shortmid \ldots \shortmid V_m)$. In general the diagonalizing matrix can be written as

$$P = \left((-1)^{i_1} V_1 \shortmid (-1)^{i_2} V_2 \shortmid \ldots \shortmid (-1)^{i_m} V_m \right),$$

with the powers i_ℓ

$$i_\ell = 1 \text{ or } 2, \text{ for } \ell = 1, \ldots, m.$$

Now one can subdivide the indices ℓ into two disjoint sets $\mathfrak{Z}$ and $\mathfrak{L}$, such that

$$\begin{cases} i_\ell = 1 \Rightarrow \ell \in \mathfrak{Z} \\ i_\ell = 2 \Rightarrow \ell \in \mathfrak{L}. \end{cases}$$

It is then straightforward to notice that

$$\mathfrak{L} \cap \mathfrak{Z} = \phi, \ \mathfrak{L} \cup \mathfrak{Z} = \{1, 2, \ldots, m\}.$$

Hence the number of possible permutations for $\mathfrak{Z}$ and $\mathfrak{L}$ is 2^m. Recall that

$$W_0 = P^T U_0 = \left(w_1^0, w_2^0, \ldots, w_m^0\right)^T.$$

Since we have 2^m different diagonalizing matrices P^T, we can write

$$W_0 = \begin{cases} w_\ell^0 = \langle V_\ell, U_0 \rangle, \ \ell \in \mathfrak{L}, \\ w_\ell^0 = \langle (-1) V_\ell, U_0 \rangle, \ \ell \in \mathfrak{Z}. \end{cases}$$

This along with (1.8) guarantees that the elements of W_0 are positive, i.e.

$$\Sigma_{\mathfrak{L},\mathfrak{Z}} = \left\{ U_0 \in \mathbb{R}^m : w_\ell^0 = \langle V_\ell, U_0 \rangle \geq 0, \quad \ell \in \mathfrak{L}, w_\ell^0 = \langle (-1) V_\ell, U_0 \rangle \geq 0, \ \ell \in \mathfrak{Z} \right\}.$$

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