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Invariant means and matrix transformations

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Abstract

In the present paper, we study the space $\ell_\infty(p,u)$ and investigate the matrix classes viz., $(\ell_\infty(p,u), v^\sigma)$ and $(\ell_\infty(p,u), v_\infty^\sigma)$, where v^σ is the space of all bounded sequences all of whose σ-means are equal, v_∞^σ is the space of all σ-bounded sequences.

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1. Introduction

Let ω denote the set of all sequences(real or complex). Any subspace of ω is called the sequence space. Let **N**, **R** and **C** denotes the set of non-negative integers, the set of real numbers and the set of complex numbers, respectively. Let ℓ_∞, c and c_0, respectively, denotes the space of all bounded sequences, the space of convergent sequences and the sequences converging to zero.

Let T denote the shift operator on ω, that is, $Tx = \{x_n\}_{n=1}^\infty$, $T^2x = \{x_n\}_{n=2}^\infty$ and so on. A Banach limit L is defined on ℓ_∞ as a non-negative linear functional such that L is invariant *i.e.*, $L(Tx) = L(x)$ and $L(e) = 1,\ e = (1, 1, 1, \ldots)$.

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Lorentz [10], called a sequence $\{x_n\}$ almost convergent if all Banach limits of x, $L(x)$, are same and this unique Banach limit is called F-limit of x. In his paper, Lorentz proved the following criterian for almost convergent sequences.
A sequence $x = \{x_n\} \in \ell_\infty$ is almost convergent with F-limit $L(x)$ if and only if

$$\lim_{m\to\infty} t_{mn}(x) = L(x)$$

where, $t_{mn}(x) = \frac{1}{m}\sum_{j=0}^{m-1} T^j x_n,\ (T^0 = 0)$ uniformly in $n \geq 0$.

We denote the set of almost convergent sequences by f.
Nanda [14] has defined a new set of sequences f_∞ as follows:

$$f_\infty = \left\{x \in \ell_\infty : \sup_{mn} |t_{mn}(x)| < \infty\right\}.$$

We call f_∞ as the set of all almost bounded sequences.

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on ℓ_∞ is said to be an invariant mean or a σ-mean if and only if (i) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n; (ii) $\phi(e) = 1$, where e={1,1,1,...}; and (iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$. Through out this paper, we deal only with mappings σ as one to one and are such that $\sigma^m(n) \neq n$, for all positive integers n and m, where $\sigma^m(n)$ denotes the mth iterate of the mapping σ at n. If σ is the translation mapping $n \to n+1$, a σ mean is often called a Banach limit (see, [1, 7]). If $x = (x_n)$, write $Tx = (Tx_n) = (x_{\sigma(n)})$. It can be shown [2] that

$$v^\sigma = \left\{\ x \in \ell_\infty : \lim_{m\to\infty} t_{mn}(x) = L \text{ uniformly in } n,\ L = \sigma - \lim x\right\},$$

where,

$$t_{mn}(x) = \frac{1}{m+1}\sum_{j=0}^{m} T^j x_n,\ T^j x_n = x_{\sigma^j(n)},\ t_{-1,n}(x) = 0.$$

We define v_∞^σ the space of σ-bounded sequences [12] in the following wa:

$$v_\infty^\sigma = \{x \in w : \sup_{m,n} |\phi_{m,n}(x)| < \infty\},$$

where,

$$\begin{aligned}\phi_{m,n}(x) &= t_{m,n}(x) - t_{m-1,n}(x)\\ &= \frac{1}{m(m+1)}\sum_{j=1}^{m} j(T^j x_n - T^{j-1}x_n).\end{aligned} \tag{1.1}$$

If $\sigma(n) = n+1$, then v_∞^σ is the set of almost bounded sequences f_∞ [6, 12, 13, 19]. The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by several authors viz., [2, 6, 12, 13]. Let $u = (u_k)$ be the sequence of non-negative real numbers. The idea of studying sequence spaces associated with multiplier sequences was introduced by Goes and Goes [8]. Later on it was follows by Savas [16, 17], Tripathy and Chandra [20], Tripathy and Hazarika [21], Tripathy and Mahanta [22] and many others. The object of this paper is to deal with the space $\ell_\infty(p,u)$ and characterize the classes of matrices $(\ell_\infty(p,u), v^\sigma)$ and $(\ell_\infty(p,u), v_\infty^\sigma)$. The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by several authors viz., [2, 5, 7, 15, 18, 23]. Thus, following, Bullet and Cakar [3], Jalal and Ahmad [9], we define the space $\ell_\infty(p,u)$ as follows:

$$\ell_\infty(p,u)=\left\{x:\sup_k |u_kx_k|^{p_k}<\infty\right\}.$$

We note that if we take $u_k=k^s$ $(s>0)$, we get the results obtained by Hamid [4]. Again if we take $u_k=k^s$ and $\sigma(n)\to n+1$, we get the result obtain by Jalal and Ahmad [9].

2. Some matrix transformations

Let X,Y be two sequence spaces and let $A=(a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n,k\in\mathbb{N}$. Then, the matrix A defines the A-transformation from X *into* Y, if for every sequence $x=(x_k)\in X$ the sequence $Ax=\{(Ax)_n\}$, the A-transform of x exists and is in Y; where $(Ax)_n=\sum\limits_k a_{nk}x_k$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $A\in(X:Y)$ we mean the characterizations of matrices from X to Y *i.e.*, $A:X\to Y$. A sequence x is said to be A-summable to l if Ax converges to l which is called as the A-limit of x.
We note that, if Ax is defined, then it follows from (1.1) that, for all integers $n,m\geq 0$

$$\phi_{m,n}(Ax)=\sum_k \wp(n,k,m)x_k$$

where

$$\wp(n,k,m)=\frac{1}{m(m+1)}\sum_{j=1}^{m} j\{a(\sigma^j(n),k)-a(\sigma^{j-1}(n),k)\}$$

Theorem 2.1. *Let* $1<p_k\leq\sup\limits_k p_k=H<\infty$ *for every* k, *then* $A\in(\ell_\infty(p,s),v_\infty^\sigma)$ *if and only if there exists an integer* $N_0>1$ *such that*

$$\sup_{m,n}\sum_k|\wp(n,k,m)|^{q_k}u^{\frac{1}{p_k}}N_0^{\frac{1}{p_k}}<\infty. \tag{2.1}$$

Proof. Let $A\in(\ell_\infty(p,u),v_\infty^\sigma)$ and that $x\in\ell_\infty(p,u)$. Put

$$q_n(x)=\sup_m|\phi_{mn}(Ax)|.$$

For $n>0$, q_n is continuous semi-norm on $\ell_\infty(p,u)$ and (q_n) is pointwise bounded on $\ell_\infty(p,u)$. Suppose that (2.1) is not true. Then there exists $x\in\ell_\infty(p,u)$ with

$$\sup_n q_n(x)=\infty.$$

By the principle of condensation of singularities [24], the set

$$\left\{x\in\ell_\infty(p,u):\sup_n q_n(x)=\infty\right\}$$

is of second category in $\ell_\infty(p,s)$ and hence nonempty i.e.,there is $x\in\ell_\infty(p,u)$ with $\sup_n q_n(x)=\infty$. But this contradicts the fact that q_n is pointwise bounded on $\ell_\infty(p,u)$. Now, by Uniform bounded principle, there is constant M such that

$$q_n(x) \leq Mg(x) \tag{2.2}$$

Applying (2.2) to the sequence $x = (x_k)$ defined as in [3] by replacing $a_{nk}(i)$ by $a(n,k,m)$, we then obtain the necessity of (2.1).

Sufficiency. We now suppose that (2.1) holds and $x \in \ell_\infty(p,u)$. Using the following inequality

$$|ab| \leq C(|a|^q C^{-q} + |b|^p)$$

for $C > 0$ and a, b two complex numbers ($p > 1$ and $p^{-1} + q^{-1} = 1$) [12, 24], we have

$$\begin{aligned}
|\phi_{m,n}(Ax)| &= \left|\sum_k \wp(n,k,m)x_k\right| \\
&\leq \sum_k |\wp(n,k,m)x_k| \\
&\leq \sum_k N_0 \left[|\wp(n,k,m)|^{q_k} u^{\frac{1}{p_k}} N_0^{\frac{1}{p_k}} + |x_k|^{p_k} u^{\frac{-1}{p_k}}\right].
\end{aligned}$$

Taking the supremum over m, n and using (2.1) we get $Ax \in v_\infty^\sigma$ for $x \in \ell_\infty(p,u)$ i.e, $A \in (\ell_\infty(p,u), v_\infty^\sigma)$. This completes the proof of the theorem. □

Theorem 2.2. *Let $1 < p_k \leq \sup_k p_k = H < \infty$ for every k, then $A \in (\ell_\infty(p,u), v^\sigma)$ if and only if there exists an integer $N_0 > 1$ such that*

$$(i) \quad \sup_{m,n} \sum_k |\wp(n,k,m)|^{q_k} u^{\frac{1}{p_k}} N_0^{\frac{1}{p_k}} < \infty,$$

$$(ii) \quad \lim_m \wp(n,k,m) = a_k \text{ uniformly in } n, \text{ for every } k.$$

Proof. **Necessity:** Let $A \in (\ell_\infty(p,u), v^\sigma)$ and that $x \in \ell_\infty(p,u)$. Let

$$q_n(x) = \sup_m |t_{mn}(Ax)|.$$

It is easy to see that for $n \geq 0$, q_n is continuous semi-norm on $\ell_\infty(p,u)$ and q_n is pointwise bounded on $\ell_\infty(p,u)$. Suppose that (i) is not true. Then, there exists $x \in \ell_\infty(p,u)$ with $\sup_n q_n(x) = \infty$. By the principle of condensation of singularities [24], the set

$$\left\{x \in l(p,u) : \sup_n q_n(x) = \infty\right\}$$

is of second category in $\ell_\infty(p,u)$ and hence non empty i.e, there exists $x \in \ell_\infty(p,u)$ with $\sup_n q_n(x) = \infty$. But this contradicts the fact that (q_n) is pointwise bounded on $\ell_\infty(p,u)$. Now by Banach-Steinhauss theorem, there is constant M such that

$$q_n(x) \leq Mg(x). \tag{2.3}$$

Now, we define a sequence $x = (x_k)$ by

$$x_k = \begin{cases} (sgn\ \wp(n,k,m))u^{\frac{1}{p_k}} N_0^{\frac{-1}{p_k}}, & 1 \leq k \leq k_0 \\ 0, & k > k_0 \end{cases}$$

Then, it is easy to see that $x \in \ell(p,u)$. Applying this sequence to (2.3) we get the condition (i). Since $e_k \in \ell_\infty(p,u)$, condition (ii) follows immediately on considering $x = e_k = (0,0,\ldots,1,0,\ldots)$, where the only 1 appears at the k-th place.

Sufficiency. We now suppose that (i) and (ii) holds and $x \in \ell_\infty(p,u)$. For $j \geq 1$

$$\sum_{k=1}^{j} |\wp(n,k,m)|^{q_k} u^{\frac{1}{p_k}} N_0^{\frac{1}{p_k}} \leq \sup_m \sum_k |t(n,k,m)|^{q_k} u^{\frac{1}{p_k}} N_0^{\frac{1}{p_k}} < \infty \text{ for every } n.$$

Therefore,

$$\sum_k |\alpha_k|^{q_k} u^{\frac{1}{p_k}} N_0^{\frac{1}{p_k}} = \lim_j \lim_m \sum_{k=1}^{j} |\wp(n,k,m)|^{q_k} u^{\frac{1}{p_k}} N_0^{\frac{1}{p_k}}$$
$$\leq \sup_m \sum_k |\wp(n,k,m)|^{q_k} u^{\frac{1}{p_k}} N_0^{\frac{1}{p_k}} < \infty.$$

Consequently the series $\sum_k \wp(n,k,m)x_k$ and $\sum_k \alpha_k x_k$ converges for every n,m and for every $x \in \ell_\infty(p,u)$. Now for $\varepsilon > 0$ and $x \in l_\infty(p,u)$, there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{k \geq k_0+1} |x_k|^{p_k} u^{\frac{-1}{p_k}} < \varepsilon.$$

By condition (ii), there exits m_0 such that

$$\left| \sum_{k=1}^{k_0} [\wp(n,k,m) - \alpha_k] \right| < \infty$$

for every $m > m_0$. By condition (i), it follows that

$$\left| \sum_{k \geq k_0+1} [\wp(n,k,m) - \alpha_k] \right|$$

is arbitrarily small. Therefore

$$\lim_m \sum_k \wp(n,k,m)x_k = \sum_k \alpha_k x_k \text{ uniformly in } n.$$

Hence $A \in (\ell_\infty(p,u), v^\sigma)$. Hence, the proof is complete. □

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