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Third-order differential subordination and superordination results by using Fox-Wright generalized hypergeometric function

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Abstract

we derive some third-order differential subordination and superordination results for some analytic p-valent functions defined in the unit disc, these results associated with Fox-Wright generalized hypergeometric function. The results are obtained by investigating appropriate classes of admissible functions. Also, sandwich-type results will be noted.

Keywords: Analytic function, multivalent function, Fox-Wright hypergeometric function, differential subordination and superordination.
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1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disc $U = \{z : z \in \mathbb{C} : |z| < 1\}$. For $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $a \in \mathbb{C}$, let $H[a, n]$ be the subclass of $H(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$. We note that $H[0, p] = H_p$.
For two functions $f(z)$ and $g(z)$, analytic in U, we say that $f(z)$ is subordinate to $g(z)$ in U, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $\omega(z)$ which (by definition) is analytic in U, satisfying the following conditions (see [3], see also [16], [17]):

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \ (z \in U),$$

such that

$$f(z) = g(\omega(z)) \ (z \in U).$$

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Indeed it is known that

$$f(z) \prec g(z) \quad (z \in U) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

In particular, If the function $g(z)$ is univalent in U, we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in U) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Let $\mathcal{A}(p)$ denotes the class of analytic functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are analytic and p-valent in the unit disc U, let $\mathcal{A}(1) = \mathcal{A}$.
For two functions $f_j \in \mathcal{A}(p) (j = 1, 2)$ are given by $f_j(z) = z^p + \sum\limits_{k=p+1}^{\infty} a_{k,j} z^k$, the Hadamard product (or convolution) of f_1 and f_2 in $\mathcal{A}(p)$ is defined by

$$\left(f_1 * f_2\right)(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k = \left(f_2 * f_1\right)(z). \tag{1.2}$$

Let $A_1, \ldots, A_q$ and $B_1, \ldots, B_s \, (q, s \in \mathbb{N})$ be non-zero real parameters, i.e. belong to $\mathbb{R}^* = \mathbb{R} \backslash \{0\}$, be such that $1 + \sum\limits_{j=1}^{s} B_j - \sum\limits_{j=1}^{q} A_j \geq 0$. Also, let the complex parameters $\alpha_1, \alpha_2, \ldots, \alpha_q$ and $\beta_1, \beta_2, \ldots, \beta_s$ be such that $\alpha_j + kA_j \neq 0, -1, -2, \ldots \, (j = 1, 2, \ldots, q;\ k = 0, 1, 2, \ldots)$ and $\beta_j + kB_j \neq 0, -1, -2, \ldots \, (j = 1, 2, \ldots, s;\ k = 0, 1, 2, \ldots)$.
Then, the Fox-Wright generalized hypergeometric function is defined for $z \in \mathbb{C}$ by the series (see [9], [24], [29] and [30])

$$\begin{aligned} {}_q\Psi_s \left[(\alpha_1, A_1), \ldots, (\alpha_q, A_q); (\beta_1, B_1), \ldots, (\beta_s, B_s); z\right] &= {}_q\Psi_s \left[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,s}; z\right] \\ &:= \sum_{k=0}^{\infty} \frac{\Gamma(\alpha_1+kA_1)\Gamma(\alpha_2+kA_2)\ldots\,\Gamma(\alpha_q+kA_q)}{\Gamma(\beta_1+kB_1)\Gamma(\beta_2+kB_2)\ldots\,\Gamma(\beta_s+kB_s)} \frac{z^k}{k!}. \\ &= \sum_{k=0}^{\infty} \frac{\prod\limits_{j=1}^{q} \Gamma(\alpha_j+kA_j)}{\prod\limits_{j=1}^{s} \Gamma(\beta_j+kB_j)} \frac{z^k}{k!} \end{aligned} \tag{1.3}$$

$$\Big(A_j \in \mathbb{R}^*, \alpha_j \in \mathbb{C}, \alpha_j + kA_j \neq 0, -1, -2, \ldots (j = 1, 2, \ldots, q;\ k = 0, 1, 2, \ldots); B_j \in \mathbb{R}^*, \beta_j \in \mathbb{C},$$

$$\beta_j + kB_j \neq 0, -1, -2, \ldots (j = 1, 2, \ldots, s;\ k = 0, 1, 2, \ldots);\ 1 + {}_{j=1}^{s} B_j - {}_{j=1}^{q} A_j \geq 0 \Big).$$

The condition $1 + {}_{j=1}^{s} B_j - {}_{j=1}^{q} A_j \geq 0$ is essential so that the series in (1.3) is absolutely convergent for all $z \in \mathbb{C}$, and is an entire function of z (for details, see [12]).
Fox-Wright generalized hypergeometric function has the following special cases of functions, defined as follows:
(i) If $A_j = 1 (j = 1, \ldots, q)$, $B_j = 1 \, (j = 1, \ldots, s)$, $q \leq s + 1$ and

$$\Upsilon := \left({}_{j=1}^{s} \Gamma(\beta_j)\right) \left({}_{j=1}^{q} \Gamma(\alpha_j)\right)^{-1}, \tag{1.4}$$

then we have the relationship:

$$\Upsilon\ {}_q\Psi_s\left[(\alpha_j,1)_{1,q};(\beta_j,1)_{1,s};z\right] = {}_qF_s\left(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z\right), \tag{1.5}$$

where ${}_qF_s\left(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z\right)$ is the generalized hypergeometric function (see [7]).
(ii) If $q=0,\ s=1,\ \beta,z\in\mathbb{C}$ and $B\in\mathbb{R}^*$, then we obtain

$$\varphi(B,\beta;z) ={}_0\ \Psi_1\left[-;(\beta,B);z\right]=\sum_{k=0}^{\infty}\frac{1}{\Gamma(\beta+kB)}\frac{z^k}{k!} \tag{1.6}$$

$(B\in\mathbb{R}^*;\beta,z\in\mathbb{C};\beta+kB\neq 0,-1,-2,\ldots(k=0,1,\ldots))$, which is known as the Wright function (see [8] and [28], Section 18.1). When $B=\delta$, $\beta=\nu+1$ and z is replaced by $-z$, then the function $\varphi(\beta,B;z)$ is denoted by $J_\nu^\delta(z)$,

$$J_\nu^\delta(z)\equiv\ {}_0\Psi_1\ \left[-;(\nu+1,\delta);-z\right]=\varphi(\delta,\nu+1;-z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\nu+1+\delta k)}\frac{z^k}{k!}, \tag{1.7}$$

which is known as the Bessel-Maitland function or the Wright generalized Bessel function (see [13], page 352 and [15], Section 8), also, for $\delta=1$, corresponds to the classical Bessel function $J_\nu(z)$.
(iii) If $q=1,\ s=1,\ \mu,z\in\mathbb{C}$ and $\lambda\in\mathbb{R}^*$, then we obtain the generalized Mittag-Leffler function $E_{\lambda,\mu}(z)$ (see [11]),

$$E_{\lambda,\mu}(z):=\ {}_1\Psi_1\ \left[(1,1);(\mu,\lambda);z\right]=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu+k\lambda)} \tag{1.8}$$

$(\lambda\in\mathbb{R}^*;\mu,z\in\mathbb{C};\mu+k\lambda\neq 0,-1,-2,\ldots(k=0,1,\ldots))$.
Other particular cases of Fox-Wright generalized hypergeometric function (1.3), were presented in [12].

Using the Wright generalized hypergeometric functions, the linear operator

$$\Theta_p\left[(\alpha_j,A_j)_{1,q};(\beta_j,B_j)_{1,s}\right]:\mathcal{A}(p)\to\mathcal{A}(p),$$

is defined by convolution, as follows (see [22]):

$$\Theta_p\left[(\alpha_j,A_j)_{1,q};(\beta_j,B_j)_{1,s}\right]f(z)=\Upsilon\left\{z^p_q\Psi_s\left[(\alpha_j,A_j)_{1,q};(\beta_j,B_j)_{1,s};z\right]\right\}*f(z), \tag{1.9}$$

where Υ is defined by (1.4).
We observe that for function $f(z)\in\mathcal{A}(p)$ defined by (1.1), we have

$$\Theta_p\left[(\alpha_j,A_j)_{1,q};(\beta_j,B_j)_{1,s}\right]f(z)=z^p+\sum_{k=p+1}^{\infty}\Upsilon\left(\vartheta_k\left[\alpha_j,\beta_j\right]\right)a_kz^k, \tag{1.10}$$

where

$$\vartheta_k\left[\alpha_j,\beta_j\right]=\frac{\Gamma(\alpha_1+A_1(k-p))\,\Gamma(\alpha_2+A_2(k-p))\ldots\ \Gamma(\alpha_q+A_q(k-p))}{\Gamma(\beta_1+B_1(k-p))\,\Gamma(\beta_2+B_2(k-p))\ldots\ \Gamma(\beta_s+B_s(k-p))\,(k-p)!}. \tag{1.11}$$

We note that for A_j=1 $(j=1,2,\ldots,q)$ and B_j=1 $(j=1,2,\ldots,s)$, we obtain the operator $H_{p,q,s}[\alpha_1]$, which was introduced and studied by Dziok and Srivastava [7]. Also for $f(z)\in\mathcal{A}$, we have the operator $\theta[\alpha_1]$ which was introduced by Dziok and Raina [6] and Aouf and Dziok [2].
Moreover, we can state the following operators as a special cases of the operator
$\Theta_p[(\alpha_j,A_j)_{1,q};(\beta_j,B_j)_{1,s}]$ defined by (1.10), for $f(z)\in\mathcal{A}(p)$, $A_j=1\,(j=1,\ldots,q)$, $B_j=1\,(j=1,\ldots,s)$, $q=2$ and $s=1$, we have:
(i) $\Theta_p\left[(a,1),(1,1);(c,1)\right]f(z)=L_p(a,c)\,f(z)\ \left(a,c\in\mathbb{C}\backslash\mathbb{Z}_0^-=\{0,-1,-2,\ldots\},p\in\mathbb{N}\right)$ (see Saitoh [21]);

(ii) $\Theta_p[(\mu+p,1),(1,1);(1,1)]\,f(z)=D^{\mu+p-1}f(z)\ (\mu>-p,\ p\in\mathbb{N})$, where $D^{\mu+p-1}f(z)$ is the $(\mu+p-1)$-the order Ruscheweyh derivative of a function $f(z)\in\mathcal{A}(p)$ see (Kumar and Shukla [14] and Goel and Sohi [10];

(iii) $\Theta_p[(1+p,1),(1,1);(1+p-\mu,1)]\,f(z)=\Omega_z^{(\mu,p)}f(z)$, where the operator $\Omega_z^{(\mu,p)}$ is defined by (see Srivastava and Aouf [23]:

$$\Omega_z^{(\mu,p)}f(z)=\frac{\Gamma(1+p-\mu)}{\Gamma(1+p)}z^{\mu}D_z^{\mu}f(z)\ \ (-\infty<\mu<p+1;p\in\mathbb{N}),$$

where D_z^{μ} is the fractional derivative operator [20];

(iv) $\Theta_p[(\nu+p,1),(1,1);(\nu+p+1,1)]\,f(z)=J_{\nu,p}(f)(z)\,(\nu>-p,\ p\in\mathbb{N})$, where $J_{\nu,p}(f)(z)$ is the generalized Bernardi-Libera-Livingston integral operator [5].

(v) $\Theta_p[(p+1,1),(1,1);(n+p,1)]\,f(z)=I_{n,p}f(z)\,(n\in\mathbb{Z},\ n>-p,\ p\in\mathbb{N})$, where the operator $I_{n,p}$ was introduced by Noor and Noor [19];

(vi) $\Theta_p[(\lambda+p,1),(c,1);(a,1)]\,f(z)=I_p^{\lambda}(a,c)\,f(z)\left(a,c\in\mathbb{R}\backslash\mathbb{Z}_0^-,\lambda>-p,\ p\in\mathbb{N}\right)$, where $I_p^{\lambda}(a,c)$ is the Cho-Kwon-Srivastava operator [4].

For convenience, we write

$$\Theta_{p,q,s}[\alpha_1]\,f(z):=\Theta_p\left[(\alpha_j,A_j)_{1,q};(\beta_j,B_j)_{1,s}\right]f(z),$$

and

$$\Theta_{p,q,s}[\beta_1]\,f(z):=\Theta_p\left[(\alpha_j,A_j)_{1,q};(\beta_j,B_j)_{1,s}\right]f(z).$$

Using (1.10), one can easily verify that

$$z\left(\Theta_{p,q,s}[\alpha_1]\,f(z)\right)'=\tfrac{\alpha_1}{A_1}\left(\Theta_{p,q,s}[\alpha_1+1]\,f(z)\right)-\tfrac{\alpha_1-pA_1}{A_1}\left(\Theta_{p.q.s}[\alpha_1]\,f(z)\right), \tag{1.12}$$

and

$$z\left(\Theta_{p,q,s}[\beta_1+1]\,f(z)\right)'=\tfrac{\beta_1}{B_1}\left(\Theta_{p,q,s}[\beta_1]\,f(z)\right)-\tfrac{\beta_1-pB_1}{B_1}\left(\Theta_{p,q,s}[\beta_1+1]\,f(z)\right). \tag{1.13}$$

2. Preliminaries

Recently, Antonino and Miller [1] (see also [25]) have extended the theory of second-order differential subordinations in U introduced by Miller and Mocanu [17] to the third-order case. They determined properties of functions p that satisfy the following third-order differential subordination:

$$\left\{\psi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z):z\in U\right\}\subset\Omega, \tag{2.1}$$

where Ω is a set in $\mathbb{C}$, p is analytic function and $\psi:\mathbb{C}^4\times U\longrightarrow\mathbb{C}$.
More recently, Tang *et al.* [27] (see also Tang *et al.* [26]) have extended the theory of second-order differential superordination in U introduced by Miller and Mocanu [18] to the third-order case. They determined properties of functions p that satisfy the following third-order differential superordination:

$$\Omega\subset\left\{\psi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z):z\in U\right\}, \tag{2.2}$$

where Ω is a set in $\mathbb{C}$, p is analytic function and $\psi:\mathbb{C}^4\times U\longrightarrow\mathbb{C}$.
In order to introduce our main results, we shall need the following definitions and lemmas:

Definition 2.1 ([1], page 440, Definition 1)**.** Let $\psi:\mathbb{C}^4\times U\longrightarrow\mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$is analytic in U and satisfies the third-order differential subordination

$$\psi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)\prec h(z), \tag{2.3}$$

then $p(z)$ is called a solution of the differential subordination. A univalent function $q(z)$ is called a dominant of the solutions of the differential subordination if $p(z)\prec q(z)$ for all $p(z)$ satisfying (2.3). A dominant $\widetilde{q}(z)$ that satisfies $\widetilde{q}(z)\prec q(z)$for all dominants $q(z)$ of (2.3) is called the best dominant.

Definition 2.2 ([27], page 3, Definition 5)**.** Let $\psi : \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ and $h(z)$ are analytic in U. If the functions $p(z)$ and $\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$ are univalent in$U$ and satisfy the following third-order differential superordination:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z), \tag{2.4}$$

then $p(z)$ is called a solution of the differential superordination. An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination, or simply a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfying (2.4). A univalent subordinant $\widetilde{q}(z)$ that satisfies $q(z) \prec \widetilde{q}(z)$ for all subordinants $q(z)$ of (2.4) is called the best subordinant.

Definition 2.3 ([1], page 441, Definition 2)**.** Denote by Q the set of all functions q that are analytic and injective on $\overline{U}\backslash E(q)$ where

$$E(q) = \left\{\xi \in \partial U : \lim_{z\to\xi} q(z) = \infty\right\}, \tag{2.5}$$

and are such that $q'(\xi) \neq 0$ for $\xi \in \partial U\backslash E(q)$. Further, let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) \equiv Q_0$.
The following classes of admissible functions will be required.

Definition 2.4 ([1], page 449, Definition 3)**.** Let Ω be a set in $\mathbb{C}$, $q \in Q$ and $n \in \mathbb{N}\backslash\{1\}$. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(r, s, t, u; z) \notin \Omega, \tag{2.6}$$

whenever

$$\begin{gathered} r = q(\xi),\ \ s = k\xi q'(\xi), \\ Re\left\{\frac{t}{s} + 1\right\} \geq kRe\left\{1 + \frac{\xi q''(\xi)}{q'(\xi)}\right\}, \\ Re\left\{\frac{u}{s}\right\} \geq k^2 Re\left\{\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right\}, \end{gathered} \tag{2.7}$$

where $z \in U$, $\xi \in \partial U\backslash E(q)$ and $k \geq n$.

Definition 2.5 ([27], page 4, Definition 7)**.** Let Ω be a set in $\mathbb{C}$, $q \in H[a, n]$ with $q'(z) \neq 0$ and $n \in \mathbb{N}\backslash\{1\}$. The class of admissible functions $\Psi_n^{'}[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(r, s, t, u; \xi) \in \Omega, \tag{2.8}$$

whenever

$$\begin{gathered} r = q(z),\ \ s = \frac{zq'(z)}{m}, \\ Re\left\{\frac{t}{s} + 1\right\} \leq \frac{1}{m} Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\}, \\ Re\left\{\frac{u}{s}\right\} \leq \frac{1}{m^2} Re\left\{\frac{z^2 q'''(z)}{q'(z)}\right\}, \end{gathered} \tag{2.9}$$

where $z \in U$, $\xi \in \partial U$ and $m \geq n$.

Lemma 2.6 ([1], Theorem 1)**.** *Let $p \in H[a, n]$with $n \geq 2$. Also let $q \in Q(a)$ and satisfies the following conditions:*

$$Re\left\{\frac{\xi q''(\xi)}{q'(\xi)}\right\} \geq 0, \qquad \left|\frac{zp'(z)}{q'(\xi)}\right| \leq k, \tag{2.10}$$

where $z \in U$, $\xi \in \partial U\backslash E(q)$ and $k \geq n$. If Ω is a set in $\mathbb{C}$, $\psi \in \Psi_n[\Omega, q]$ and

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega, \tag{2.11}$$

then

$$p(z) \prec q(z) \ \ (z \in U)\,. \tag{2.12}$$

Lemma 2.7 ([27], Theorem 8). *Let* $\psi \in \Psi'_n[\Omega, q]$. *If* $\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$ *is univalent in* U, $p \in Q(a)$ *and* $q \in H[a, n]$*satisfy the following conditions:*

$$Re\left\{\frac{\xi q''(\xi)}{q'(\xi)}\right\} \geq 0, \qquad \left|\frac{zp'(z)}{q'(\xi)}\right| \leq m, \tag{2.13}$$

where $z \in U$, $\xi \in \partial U$ *and* $m \geq n \geq 2$, *then*

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) : z \in U\right\}, \tag{2.14}$$

implies that

$$q(z) \prec p(z) \ \ (z \in U)\,. \tag{2.15}$$

In the next two sections, by making use of the third-order differential subordination results of Antonino and Miller [1] in the unit disk U and the third-order differential superordination results in U obtained by Tang et al. [27] (see also Tang *et al.* [26]), we determine certain appropriate classes of admissible functions and investigate some third-order differential subordination and differential superordination properties of meromorphically multivalent functions associated with the operator $\Theta_{p,q,s}\left[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,s}\right]$ defined by (1.10).

3. Third-order differential subordination results

For convenience, unless otherwise mentioned, we shall assume throughout the paper that $A_j \in \mathbb{R}^*$ and $\alpha_j \in \mathbb{C}$ be such that $\alpha_j + kA_j \neq 0, -1, -2, \ldots$ ($j = 1, 2, \ldots, q$; $k = 0, 1, 2, \ldots$). Also we assume that $B_j \in \mathbb{R}^*$ and $\beta_j \in \mathbb{C}$ be such that $\beta_j + kB_j \neq 0, -1, -2, \ldots$ ($j = 1, 2, \ldots, s$; $k = 0, 1, 2, \ldots$), moreover, $1 + {}_{j=1}^{s} B_j - {}_{j=1}^{q} A_j \geq 0$ and $z \in U$.
In this section, we obtain some third-order differential subordination results. For this aim, the class of admissible functions is defined as follows:

Definition 3.1. Let Ω be a set in $\mathbb{C}$ and $q \in Q_0 \cap H_p$. The class of admissible functions $\Phi_\Theta[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(a, b, c, d; z) \notin \Omega, \tag{3.1}$$

whenever

$$a = q(\xi), \quad b = \frac{n\xi q'(\xi) + \frac{\beta_1 - pB_1}{B_1} q(\xi)}{\frac{\beta_1}{B_1}}, \tag{3.2}$$

$$Re\left\{\frac{\beta_1(\beta_1-1)c - (\beta_1-pB_1)(\beta_1-pB_1-1)a}{B_1[\beta_1 b - (\beta_1-pB_1)a]} - \frac{2(\beta_1-pB_1)-1}{B_1}\right\} \geq nRe\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\}, \tag{3.3}$$

and

$$Re\left\{\frac{\beta_1(\beta_1-1)(\beta_1-2)d + (\beta_1-pB_1)(\beta_1-pB_1-1)(2(\beta_1-pB_1)+3B_1-1)a - 3\beta_1(\beta_1-1)((\beta_1-pB_1)+B_1-1)c}{B_1^2[\beta_1 b - (\beta_1-pB_1)a]}\right.$$
$$\left. + \frac{3(\beta_1-pB_1)(\beta_1-pB_1+2B_1-3)+(B_1-1)(4B_1-5)}{B_1^2}\right\} \geq n^2 Re\left\{\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right\}, \tag{3.4}$$

where $n \in \mathbb{N} \backslash \{1\}$, $\xi \in \partial U \backslash E(q)$and $z \in U$.

Theorem 3.2. *Let* $\phi \in \Phi_\Theta[\Omega, q]$. *If the functions* $f \in \mathcal{A}(p)$ *and* $q \in Q_0 \cap H_p$ *satisfy the following conditions:*

$$Re\left\{\frac{\xi q''(\xi)}{q'(\xi)}\right\}\geq 0,$$
$$\left|\beta_1\Theta_{p,q,s}\left[\beta_1\right]f\left(z\right)-\left(\beta_1-pB_1\right)\Theta_{p,q,s}\left[\beta_1+1\right]f\left(z\right)\right|\leq n\left|B_1q'(\xi)\right|. \tag{3.5}$$

If

$$\left\{\phi\left(\Theta_{p,q,s}\left[\beta_1+1\right]f\left(z\right),\Theta_{p,q,s}\left[\beta_1\right]f\left(z\right),\Theta_{p,q,s}\left[\beta_1-1\right]f\left(z\right),\Theta_{p,q,s}\left[\beta_1-2\right]f\left(z\right);z\right):z\in U\right\}\subset\Omega \tag{3.6}$$

$$\left(n\in\mathbb{N}\backslash\left\{1\right\},\ \xi\in\partial U\backslash E(q)\ \textit{and}\ z\in U\right),$$

then

$$\Theta_{p,q,s}\left[\beta_1+1\right]f\left(z\right)\prec q(z)\ \ (z\in U). \tag{3.7}$$

Proof. Define the analytic function $g(z)$ by

$$g(z)=\Theta_{p,q,s}\left[\beta_1+1\right]f\left(z\right). \tag{3.8}$$

Making use of (1.13) and (3.8), we have

$$\Theta_{p,q,s}\left[\beta_1\right]f\left(z\right)=\frac{zg'(z)+\frac{\beta_1-pB_1}{B_1}g(z)}{\frac{\beta_1}{B_1}}. \tag{3.9}$$

Further computations shows that

$$\Theta_{p,q,s}\left[\beta_1-1\right]f\left(z\right)=\frac{z^2g''(z)+\left(1+\frac{2(\beta_1-pB_1)-1}{B_1}\right)zg'(z)+\frac{(\beta_1-pB_1)(\beta_1-pB_1-1)}{B_1^2}g(z)}{\frac{(\beta_1)(\beta_1-1)}{B_1^2}}, \tag{3.10}$$

and

$$\begin{aligned}\Theta_{p,q,s}\left[\beta_1-2\right]f\left(z\right)=&\left(z^3g'''(z)+3\left(1+\frac{\beta_1-pB_1-1}{B_1}\right)z^2g''(z)+\left(1+\frac{2-3B_1+3(\beta_1-pB_1)(\beta_1-(p-1)B_1)}{B_1^2}\right)zg'(z)\right.\\&\left.+\frac{(\beta_1-pB_1)(\beta_1-pB_1-1)(\beta_1-pB_1-2)}{B_1^3}g(z)\Big/\left(\frac{(\beta_1)(\beta_1-1)(\beta_1-2)}{B_1^3}\right)\right..\end{aligned} \tag{3.11}$$

We now define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$a(r,s,t,u)=r,\qquad b(r,s,t,u)=\frac{s+\frac{\beta_1-pB_1}{B_1}r}{\frac{\beta_1}{B_1}}, \tag{3.12}$$

$$c(r,s,t,u)=\frac{t+\left(1+\frac{2(\beta_1-pB_1)-1}{B_1}\right)s+\frac{(\beta_1-pB_1)(\beta_1-pB_1-1)}{B_1^2}r}{\frac{(\beta_1)(\beta_1-1)}{B_1^2}}, \tag{3.13}$$

and

$$\begin{aligned}d(r,s,t,u)=&\left(u+3\left(1+\frac{\beta_1-pB_1-1}{B_1}\right)t+\left(1+\frac{2-3B_1+3(\beta_1-pB_1)(\beta_1-(p-1)B_1)}{B_1^2}\right)s\right.\\&\left.+\frac{(\beta_1-pB_1)(\beta_1-pB_1-1)(\beta_1-pB_1-2)}{B_1^3}r\right)\Big/\left(\frac{(\beta_1)(\beta_1-1)(\beta_1-2)}{B_1^3}\right).\end{aligned} \tag{3.14}$$

Let

$$\psi(r,s,t,u;z)=\phi(a,b,c,d;z)$$
$$=\phi\left(t,\frac{s+\frac{\beta_1-pB_1}{B_1}r}{\frac{\beta_1}{B_1}},\frac{t+\left(1+\frac{2(\beta_1-pB_1)-1}{B_1}\right)s+\frac{(\beta_1-pB_1)(\beta_1-pB_1-1)}{B_1^2}r}{\frac{(\beta_1)(\beta_1-1)}{B_1^2}},\right.$$
$$\left.\frac{u+3\left(1+\frac{\beta_1-pB_1-1}{B_1}\right)t+\left(1+\frac{2-3B_1+3(\beta_1-pB_1)(\beta_1-(p-1)B_1)}{B_1^2}\right)s+\frac{(\beta_1-pB_1)(\beta_1-pB_1-1)(\beta_1-pB_1-2)}{B_1^3}r}{\frac{(\beta_1)(\beta_1-1)(\beta_1-2)}{B_1^3}};z\right). \tag{3.15}$$

Using Lemma 2.6, (3.8)–(3.11) and (3.12)–(3.15), we have

$$\psi\left(g(z),zg'(z),z^2g''(z),z^3g'''(z);z\right)$$
$$=\phi\left(\Theta_{p,q,s}[\beta_1{+}1]f(z),\Theta_{p,q,s}[\beta_1]f(z),\Theta_{p,q,s}[\beta_1{-}1]f(z),\Theta_{p,q,s}[\beta_1{-}2]f(z);z\right). \tag{3.16}$$

Hence, (3.6) leads to

$$\psi\left(g(z),zg'(z),z^2g''(z),z^3g'''(z);z\right)\in\Omega. \tag{3.17}$$

Moreover, using (3.12)–(3.14) and some calculations, we get

$$\frac{t}{s}+1=\frac{\beta_1(\beta_1-1)c-(\beta_1-pB_1)(\beta_1-pB_1-1)a}{B_1[\beta_1 b-(\beta_1-pB_1)a]}-\frac{2(\beta_1-pB_1)-1}{B_1}, \tag{3.18}$$

and

$$\frac{u}{s}=\frac{\beta_1(\beta_1-1)(\beta_1-2)d+(\beta_1-pB_1)(\beta_1-pB_1-1)(2(\beta_1-pB_1)+3B_1-1)a-3\beta_1(\beta_1-1)((\beta_1-pB_1)+B_1-1)c}{B_1^2[\beta_1 b-(\beta_1-pB_1)a]}$$
$$+\frac{3(\beta_1-pB_1)(\beta_1-pB_1+2B_1-3)+(B_1-1)(4B_1-5)}{B_1^2}. \tag{3.19}$$

Thus, the admissibility condition of $\phi\in\Phi_\Theta[\Omega,q]$ in Definition 3.1 is equivalent to the admissibility condition of $\psi\in\Psi_n[\Omega,q]$ as given in Definition 2.4. Therefore, by using (3.5), (3.6) and Lemma 2.6, we have $g(z)\prec q(z)$ $(z\in U)$ or equivalently $\Theta_{p,q,s}[\beta_1{+}1]f(z)\prec q(z)$ $(z\in U)$. The proof of Theorem 3.2 is thus completed. □

Using the same arguments used in [5], Corollary 2.3 b.1, page 30, our next result is an extension of Theorem 3.2 to the case when the behavior of q on ∂U is not known.

Corollary 3.3. *Let $\Omega\subset\mathbb{C}$ and let q be univalent function in U with $q(0)=0$. Let $\phi\in\Phi_\Theta[\Omega,q_\rho]$ for some $\rho\in(0,1)$ where $q_\rho(z)=q(\rho z)$. If the functions $f\in\mathcal{A}(p)$ and q_ρ satisfy the following conditions:*

$$Re\left\{\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\right\}\geq 0,$$
$$\left|\beta_1\Theta_{p,q,s}[\beta_1]f(z)-(\beta_1-pB_1)\Theta_{p,q,s}[\beta_1{+}1]f(z)\right|\leq n\left|B_1q_\rho'(\xi)\right|. \tag{3.20}$$

If

$$\phi\left(\Theta_{p,q,s}[\beta_1{+}1]f(z),\Theta_{p,q,s}[\beta_1]f(z),\Theta_{p,q,s}[\beta_1{-}1]f(z),\Theta_{p,q,s}[\beta_1{-}2]f(z);z\right)\in\Omega, \tag{3.21}$$

then

$$\Theta_{p,q,s}[\beta_1{+}1]f(z)\prec q(z)$$
$$(n\in\mathbb{N}\backslash\{1\},\ \xi\in\partial U\backslash E(q)\ \text{and}\ z\in U).$$

Proof. As a consequence of Theorem 3.2, we have

$$\Theta_{p,q,s}[\beta_1{+}1]\, f(z) \prec q_\rho(z). \tag{3.22}$$

Now, the proof of Corollary 3.3 can be deduced from the following subordination property:

$$q_\rho(z) \prec q(z). \tag{3.23}$$

The proof of Corollary 3.3 is thus completed. □

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\Phi_\Theta[h(U), q]$ is written as $\Phi_\Theta[h, q]$. The following two results are immediate consequences of Theorem 3.2 and Corollary 3.3.

Theorem 3.4. *Let $\phi \in \Phi_\Theta[h, q]$. If the functions $f \in \mathcal{A}(p)$ and $q \in Q_0$ satisfy the following conditions:*

$$Re\left\{\frac{\xi q''(\xi)}{q'(\xi)}\right\} \geq 0,$$

$$\left|\beta_1\Theta_{p,q,s}[\beta_1]\, f(z) - (\beta_1{-}pB_1)\,\Theta_{p,q,s}[\beta_1{+}1]\, f(z)\right| \leq n\left|B_1 q'(\xi)\right|. \tag{3.24}$$

If

$$\phi\Big(\Theta_{p,q,s}[\beta_1{+}1]\, f(z), \Theta_{p,q,s}[\beta_1]\, f(z), \Theta_{p,q,s}[\beta_1{-}1]\, f(z), \Theta_{p,q,s}[\beta_1{-}2]\, f(z); z\Big) \prec h(z), \tag{3.25}$$

then

$$\Theta_{p,q,s}[\beta_1{+}1]\, f(z) \prec q(z)$$

$$(n \in \mathbb{N}\backslash\{1\}, \xi \in \partial U\backslash E(q) \text{ and } z \in U).$$

Corollary 3.5. *Let $\Omega \subset \mathbb{C}$ and let q be univalent function in U with $q(0) = 0$. Let $\phi \in \Phi_\Theta[h, q_\rho]$ for some $\rho \in (0,1)$ where $q_\rho(z) = q(\rho z)$. If the functions $f \in \mathcal{A}(p)$ and q_ρ satisfy the following conditions:*

$$Re\left\{\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\right\} \geq 0,$$

$$\left|\beta_1\Theta_{p,q,s}[\beta_1]\, f(z) - (\beta_1{-}pB_1)\,\Theta_{p,q,s}[\beta_1{+}1]\, f(z)\right| \leq n\left|B_1 q_\rho'(\xi)\right|. \tag{3.26}$$

If

$$\phi\Big(\Theta_{p,q,s}[\beta_1{+}1]\, f(z), \Theta_{p,q,s}[\beta_1]\, f(z), \Theta_{p,q,s}[\beta_1{-}1]\, f(z), \Theta_{p,q,s}[\beta_1{-}2]\, f(z); z\Big) \prec h(z), \tag{3.27}$$

then

$$\Theta_{p,q,s}[\beta_1{+}1]\, f(z) \prec q(z) \tag{3.28}$$

$$(n \in \mathbb{N}\backslash\{1\}, \xi \in \partial U\backslash E(q) \text{ and } z \in U).$$

Our next theorem yields the best dominant of the differential subordination (3.6) or (3.25).

Theorem 3.6. *Let the function $h(z)$ be univalent in U. Also, let $\phi\mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ and ψ be given by (3.15). Suppose that the differential equation*

$$\psi\left(q(z), zq'(z), z^2q''(z), z^3q'''(z); z\right) = h(z), \tag{3.29}$$

has a solution $q(z) \in Q_0 \cap H_p$, which satisfies the conditions in (3.5). If the function $f \in \mathcal{A}(p)$ satisfies condition (3.20) and the function

$$\phi\Big(\Theta_{p,q,s}[\beta_1{+}1]\, f(z), \Theta_{p,q,s}[\beta_1]\, f(z), \Theta_{p,q,s}[\beta_1{-}1]\, f(z), \Theta_{p,q,s}[\beta_1{-}2]\, f(z); z\Big),$$

is analytic in U, then

$$\Theta_{p,q,s}[\beta_1{+}1]\, f(z) \prec q(z), \tag{3.30}$$

and $q(z)$ is the best dominant.

Proof. By applying Theorem 3.2, we deduce that q is a dominant of (3.25). Since q satisfies (3.29), it is also a solution of (3.25). Therefore, q will be dominated by all dominants. Hence q is the best dominant. □

Next, we introduce a new admissible class, $\widetilde{\Phi}_{\Theta}[\Omega, q]$, as follows:

Definition 3.7. Let Ω be a set in $\mathbb{C}$ and $q \in Q_0 \cap H_p$. The class of admissible functions $\widetilde{\Phi}_{\Theta}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times U \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(a, b, c, d; z) \notin \Omega, \tag{3.31}$$

whenever

$$a = q(\xi), \quad b = \frac{n\xi q'(\xi) + \frac{\alpha_1 - pA_1 - 1}{A_1} q(\xi)}{\frac{\alpha_1 - 1}{A_1}}, \tag{3.32}$$

$$Re\left\{\frac{\alpha_1(\alpha_1-1)c-(\alpha_1-pA_1)(\alpha_1-pA_1-1)a}{A_1[(\alpha_1-1)b-(\alpha_1-pA_1-1)a]} - \frac{2(\alpha_1-pA_1)-1}{A_1}\right\} \geq nRe\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\}, \tag{3.33}$$

and

$$Re\left\{\frac{\alpha_1(\alpha_1-1)(\alpha_1+1)d+(\alpha_1-pA_1)(\alpha_1-pA_1-1)(3(\alpha_1-(p-1)A_1)-(\alpha_1-pA_1+1))a-3\alpha_1(\alpha_1-1)(\alpha_1-(p-1)A_1)c}{A_1[(\alpha_1-1)b-(\alpha_1-pA_1-1)a]}\right.$$
$$\left.+\frac{3(\alpha_1-pA_1)(\alpha_1-(p-1)A_1)+(A_1-1)(3(\alpha_1-pA_1)+2A_1-1)}{A_1^2}\right\} \geq n^2 Re\left\{\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right\}, \tag{3.34}$$

where $n \in \mathbb{N}\backslash\{1\}$, $\xi \in \partial U \backslash E(q)$ and $z \in U$.

Theorem 3.8. *Let $\phi \in \widetilde{\Phi}_{\Theta}[\Omega, q]$. If the functions $f \in \mathcal{A}(p)$ and $q \in Q_0 \cap H_p$ satisfy the following conditions:*

$$Re\left\{\frac{\xi q''(\xi)}{q'(\xi)}\right\} \geq 0,$$
$$\left|(\alpha_1 - 1)\Theta_{p,q,s}[\alpha_1] f(z) - (\alpha_1 - pA_1 - 1)\Theta_{p,q,s}[\alpha_1 - 1] f(z)\right| \leq n\left|A_1 q'(\xi)\right|. \tag{3.35}$$

If

$$\left\{\phi\left(\Theta_{p,q,s}[\alpha_1{-}1] f(z), \Theta_{p,q,s}[\alpha_1] f(z), \Theta_{p,q,s}[\alpha_1{+}1] f(z), \Theta_{p,q,s}[\alpha_1{+}2] f(z); z\right) sl{:}z{\in}U\right\} \subset \Omega \tag{3.36}$$

$(n \in \mathbb{N}\backslash\{1\},\ \xi \in \partial U \backslash E(q)$ *and* $z \in U)$, *then*

$$\Theta_{p,q,s}[\alpha_1{-}1] f(z) \prec q(z) \ (z \in U). \tag{3.37}$$

Proof. Define the analytic function $g(z)$ by

$$g(z) = \Theta_{p,q,s}[\alpha_1{-}1] f(z). \tag{3.38}$$

Making use of (1.12) and (3.38), we have

$$\Theta_{p,q,s}[\alpha_1] f(z) = \frac{zg'(z) + \frac{\alpha_1 - pA_1 - 1}{A_1} g(z)}{\frac{\alpha_1 - 1}{A_1}}. \tag{3.39}$$

Further computations shows that

$$\Theta_{p,q,s}[\alpha_1{+}1] f(z) = \frac{z^2 g''(z) + \left(1 + \frac{2(\alpha_1 - pA_1) - 1}{A_1}\right) zg'(z) + \frac{(\alpha_1 - pA_1)(\alpha_1 - pA_1 - 1)}{A_1^2} g(z)}{\frac{\alpha_1(\alpha_1 - 1)}{A_1^2}}, \tag{3.40}$$

and

$$\Theta_{p,q,s}\left[\alpha_1+2\right]f\left(z\right) = \frac{z^3g'''(z)+3\left(1+\frac{\alpha_1-pA_1}{A_1}\right)z^2g''(z)+\left(1+\frac{3(\alpha_1-pA_1)(\alpha_1-(p-1)A_1)-1}{A_1^2}\right)zg'(z)+\frac{(\alpha_1-pA_1)(\alpha_1-pA_1-1)(\alpha_1-pA_1+1)}{A_1^3}g(z)}{\frac{\alpha_1(\alpha_1-1)(\alpha_1+1)}{A_1^3}}. \quad (3.41)$$

We now define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$a(r,s,t,u)=r, \qquad b(r,s,t,u)=\frac{s+\frac{\alpha_1-pA_1-1}{A_1}r}{\frac{\alpha_1-1}{A_1}}, \quad (3.42)$$

$$c(r,s,t,u)=\frac{t+\left(1+\frac{2(\alpha_1-pA_1)-1}{A_1}\right)s+\frac{(\alpha_1-pA_1)(\alpha_1-pA_1-1)}{A_1^2}r}{\frac{\alpha_1(\alpha_1-1)}{A_1^2}}, \quad (3.43)$$

and

$$d(r,s,t,u)=\frac{u+3\left(1+\frac{\alpha_1-pA_1}{A_1}\right)t+\left(1+\frac{3(\alpha_1-pA_1)(\alpha_1-(p-1)A_1)-1}{A_1^2}\right)s+\frac{(\alpha_1-pA_1)(\alpha_1-pA_1-1)(\alpha_1-pA_1+1)}{A_1^3}r}{\frac{\alpha_1(\alpha_1-1)(\alpha_1+1)}{A_1^3}}. \quad (3.44)$$

Let

$$\begin{aligned}\psi\left(r,s,t,u;z\right)&=\phi\left(a,b,c,d;z\right)\\ &=\phi\left(t,\frac{s+\frac{\alpha_1-pA_1-1}{A_1}r}{\frac{\alpha_1-1}{A_1}},\frac{t+\left(1+\frac{2(\alpha_1-pA_1)-1}{A_1}\right)s+\frac{(\alpha_1-pA_1)(\alpha_1-pA_1-1)}{A_1^2}r}{\frac{\alpha_1(\alpha_1-1)}{A_1^2}},\right.\\ &\left.\frac{u+3\left[1+\frac{\alpha_1-pA_1}{A_1}\right]t+\left[1+\frac{3(\alpha_1-pA_1)(\alpha_1-(p-1)A_1)-1}{A_1^2}\right]s+\frac{(\alpha_1-pA_1)(\alpha_1-pA_1-1)(\alpha_1-pA_1+1)}{A_1^3}r}{\frac{\alpha_1(\alpha_1-1)(\alpha_1+1)}{A_1^3}};z\right).\end{aligned} \quad (3.45)$$

Using Lemma 2.6, (3.38)–(3.41) and (3.42)–(3.45), we have

$$\begin{aligned}&\psi\left(g(z),zg'(z),z^2g''(z),z^3g'''(z);z\right)\\ &=\phi\left(\Theta_{p,q,s}\left[\alpha_1-1\right]f\left(z\right),\Theta_{p,q,s}\left[\alpha_1\right]f\left(z\right),\Theta_{p,q,s}\left[\alpha_1+1\right]f\left(z\right),\Theta_{p,q,s}\left[\alpha_1+2\right]f\left(z\right);z\right).\end{aligned} \quad (3.46)$$

Hence, (3.36) implies

$$\psi\left(g(z),zg'(z),z^2g''(z),z^3g'''(z);z\right)\in\Omega. \quad (3.47)$$

Using (3.42)–(3.44), then we have

$$\frac{t}{s}+1=\frac{\alpha_1(\alpha_1-1)c-(\alpha_1-pA_1)(\alpha_1-pA_1-1)a}{A_1[(\alpha_1-1)b-(\alpha_1-pA_1-1)a]}-\frac{2(\alpha_1-pA_1)-1}{A_1}, \quad (3.48)$$

$$\begin{aligned}\frac{u}{s}\;=\;&\frac{\alpha_1(\alpha_1-1)(\alpha_1+1)d+(\alpha_1-pA_1)(\alpha_1-pA_1-1)(3(\alpha_1-(p-1)A_1)-(\alpha_1-pA_1+1))a-3\alpha_1(\alpha_1-1)(\alpha_1-(p-1)A_1)c}{A_1[(\alpha_1-1)b-(\alpha_1-pA_1-1)a]}\\ &+\frac{3(\alpha_1-pA_1)(\alpha_1-(p-1)A_1)+(A_1-1)(3(\alpha_1-pA_1)+2A_1-1)}{A_1^2}.\end{aligned} \quad (3.49)$$

Thus, the admissibility condition for $\phi\in\widetilde{\Phi}_{\Theta}[\Omega,q]$ in Definition 3.7 is equivalent to the admissibility condition for $\psi\in\Psi_n[\Omega,q]$ as given in Definition 2.4. Therefore, by using (3.35) and Lemma 2.6, we have $g(z)\prec q(z)$ $(z\in U)$ or equivalently $\Theta_{p,q,s}\left[\alpha_1-1\right]f\left(z\right)\prec q(z)$ $(z\in U)$. The proof of Theorem 3.8 is thus completed. □

Similarly, using the same arguments used in [9], Corollary 2.3 b.1, page 30, our next result is an extension of Theorem 3.8 to the case when the behavior of q on ∂U is not known.

Corollary 3.9. *Let $\Omega \subset \mathbb{C}$ and let q be univalent function in U with $q(0) = 0$. Let $\phi \in \widetilde{\Phi}_{\Theta}[\Omega, q_\rho]$ for some $\rho \in (0,1)$ where $q_\rho(z) = q(\rho z)$. If the functions $f \in \mathcal{A}(p)$ and q_ρ satisfy the following conditions:*

$$Re\left\{\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\right\} \geq 0,$$

$$\left|(\alpha_1 - 1)\,\Theta_{p,q,s}[\alpha_1{+}1]\,f(z) - (\alpha_1 - pA_1 - 1)\,\Theta_{p,q,s}[\alpha_1]\,f(z)\right| \leq n\left|A_1 q_\rho'(\xi)\right|. \tag{3.50}$$

If

$$\phi\left(\Theta_{p,q,s}[\alpha_1{-}1]\,f(z), \Theta_{p,q,s}[\alpha_1]\,f(z), \Theta_{p,q,s}[\alpha_1{+}1]\,f(z), \Theta_{p,q,s}[\alpha_1{+}2]\,f(z); z\right) \in \Omega, \tag{3.51}$$

then

$$\Theta_{p,q,s}[\alpha_1{-}1]\,f(z) \prec q(z)$$

$(n \in \mathbb{N}\backslash\{1\},\ \xi \in \partial U\backslash E(q)$ *and* $z \in U)$.

Proof. As a consequence of Theorem 3.8, we have

$$\Theta_{p,q,s}[\alpha_1{-}1]\,f(z) \prec q_\rho(z). \tag{3.52}$$

Now, the proof of Corollary 3.9 can be deduced from the following subordination property:

$$q_\rho(z) \prec q(z). \tag{3.53}$$

The proof of Corollary 3.9 is thus completed. □

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case the class $\widetilde{\Phi}_{\Theta}[h(U), q]$ is written as $\widetilde{\Phi}_{\Theta}[h, q]$. The following two results are immediate consequences of Theorem 3.8 and Corollary 3.9.

Theorem 3.10. *Let $\phi \in \widetilde{\Phi}_{\Theta}[h, q]$. If the functions $f \in \mathcal{A}(p)$ and $q \in Q_0$ satisfy the following conditions:*

$$Re\left\{\frac{\xi q''(\xi)}{q'(\xi)}\right\} \geq 0,$$

$$\left|(\alpha_1 - 1)\,\Theta_{p,q,s}[\alpha_1{+}1]\,f(z) - (\alpha_1 - pA_1 - 1)\,\Theta_{p,q,s}[\alpha_1]\,f(z)\right| \leq n\left|A_1 q'(\xi)\right|. \tag{3.54}$$

If

$$\phi\left(\Theta_{p,q,s}[\alpha_1{-}1]\,f(z), \Theta_{p,q,s}[\alpha_1]\,f(z), \Theta_{p,q,s}[\alpha_1{+}1]\,f(z), \Theta_{p,q,s}[\alpha_1{+}2]\,f(z); z\right) \prec h(z), \tag{3.55}$$

then

$$\Theta_{p,q,s}[\alpha_1{-}1]\,f(z) \prec q(z)$$

$(n \in \mathbb{N}\backslash\{1\}, \xi \in \partial U\backslash E(q)$ *and* $z \in U)$.

Corollary 3.11. *Let $\Omega \subset \mathbb{C}$ and let q be univalent function in U with $q(0) = 0$. Let $\phi \in \widetilde{\Phi}_{\Theta}[h, q_\rho]$ for some $\rho \in (0,1)$ where $q_\rho(z) = q(\rho z)$. If the functions $f \in A(p)$ and q_ρ satisfy the following conditions:*

$$Re\left\{\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\right\} \geq 0,$$

$$\left|(\alpha_1 - 1)\,\Theta_{p,q,s}[\alpha_1{+}1]\,f(z) - (\alpha_1 - pA_1 - 1)\,\Theta_{p,q,s}[\alpha_1]\,f(z)\right| \leq n\left|A_1 q_\rho'(\xi)\right|. \tag{3.56}$$

If

$$\phi\left(\Theta_{p,q,s}[\alpha_1{-}1]\,f(z), \Theta_{p,q,s}[\alpha_1]\,f(z), \Theta_{p,q,s}[\alpha_1{+}1]\,f(z), \Theta_{p,q,s}[\alpha_1{+}2]\,f(z); z\right) \prec h(z), \tag{3.57}$$

then

$$\Theta_{p,q,s}[\alpha_1{-}1]\,f(z) \prec q(z) \tag{3.58}$$

$(n \in \mathbb{N}\backslash\{1\}, \xi \in \partial U\backslash E(q)$ *and* $z \in U)$.

Our next theorem yields the best dominant of the differential subordination (3.36) or (3.55).

Theorem 3.12. *Let the function $h(z)$ be univalent in U. Also, let $\phi : \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ and ψ be given by* (3.45)*. Suppose that the differential equation*

$$\psi\left(q(z), zq'(z), z^2q''(z), z^3q'''(z); z\right) = h(z), \tag{3.59}$$

has a solution $q(z) \in Q_0 \cap H_p$, which satisfies the conditions in (3.35)*. If the function $f \in \mathcal{A}(p)$ satisfies condition* (3.51) *and the function*

$$\phi\left(\Theta_{p,q,s}\left[\alpha_1{-}1\right] f(z), \Theta_{p,q,s}\left[\alpha_1\right] f(z), \Theta_{p,q,s}\left[\alpha_1{+}1\right] f(z), \Theta_{p,q,s}\left[\alpha_1{+}2\right] f(z); z\right),$$

is analytic in U, then

$$\Theta_{p,q,s}\left[\alpha_1{-}1\right] f(z) \prec q(z), \tag{3.60}$$

and $q(z)$ is the best dominant.

Proof. By applying Theorem 3.8, we deduce that q is a dominant of (3.55). Since q satisfies (3.59), it is also a solution of (3.55). Therefore, q will be dominated by all dominants. Hence q is the best dominant. □

4. Third-order differential superordination results

In this section, we obtain some third-order differential superordination results. Also, for this purpose, the class of admissible functions is defined as follows:

Definition 4.1. Let Ω be a set in $\mathbb{C}$, $q \in H_p$ with $q'(z) \neq 0$ and $m \in \mathbb{N}\backslash\{1\}$. The class of admissible functions $\Phi'_{\Theta}[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(a, b, c, d; \xi) \in \Omega \tag{4.1}$$

whenever

$$a = q(z), \quad b = \frac{zq'(z)+m\frac{\beta_1-pB_1}{B_1}q(z)}{m\frac{\beta_1}{B_1}}, \tag{4.2}$$

$$Re\left\{\frac{\beta_1(\beta_1-1)c-(\beta_1-pB_1)(\beta_1-pB_1-1)a}{B_1[\beta_1 b-(\beta_1-pB_1)a]} - \frac{2(\beta_1-pB_1)-1}{B_1}\right\} \leq \frac{1}{m}Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}, \tag{4.3}$$

and

$$Re\left\{\frac{\beta_1(\beta_1-1)(\beta_1-2)d+(\beta_1-pB_1)(\beta_1-pB_1-1)(2(\beta_1-pB_1)+3B_1-1)a-3\beta_1(\beta_1-1)((\beta_1-pB_1)+B_1-1)c}{B_1^2[\beta_1 b-(\beta_1-pB_1)a]}\right.$$
$$\left. + \frac{3(\beta_1-pB_1)(\beta_1-pB_1+2B_1-3)+(B_1-1)(4B_1-5)}{B_1^2}\right\} \leq \frac{1}{m^2}Re\left\{\frac{z^2q'''(z)}{q'(z)}\right\}, \tag{4.4}$$

where $m \geq 2$, $\xi \in \partial U$ and $z \in U$.

Theorem 4.2. *Let $\phi \in \Phi'_{\Theta}[\Omega, q]$. If the functions $f \in \mathcal{A}(p)$ and $\Theta_{p,q,s}\left[\beta_1{+}1\right] f(z) \in Q_0$ satisfy the following conditions:*

$$Re\left\{\frac{zq''(z)}{q'(z)}\right\} \geq 0,$$

$$\left|\beta_1\Theta_{p,q,s}\left[\beta_1\right] f(z) - (\beta_1-pB_1)\,\Theta_{p,q,s}\left[\beta_1{+}1\right] f(z)\right| \leq m\left|B_1 q'(z)\right|, \tag{4.5}$$

and

$$\phi\left(\Theta_{p,q,s}\left[\beta_1{+}1\right] f(z), \Theta_{p,q,s}\left[\beta_1\right] f(z), \Theta_{p,q,s}\left[\beta_1{-}1\right] f(z), \Theta_{p,q,s}\left[\beta_1{-}2\right] f(z); z\right),$$

is univalent in U. Then

$$\Omega \subset \left\{\phi\left(\Theta_{p,q,s}\left[\beta_1{+}1\right] f(z), \Theta_{p,q,s}\left[\beta_1\right] f(z), \Theta_{p,q,s}\left[\beta_1{-}1\right] f(z), \Theta_{p,q,s}\left[\beta_1{-}2\right] f(z); z\right) : z \in U\right\}, \tag{4.6}$$

implies

$$q(z) \prec \Theta_{p,q,s}\left[\beta_1{+}1\right] f(z) \quad (z \in U). \tag{4.7}$$

Proof. Let the function $g(z)$ be defined by (3.8) and ψ be defined by (3.15). Since $\phi \in \Phi'_{\Theta}[\Omega, q]$, (3.16) and (4.6) yield

$$\Omega \subset \left\{\psi\left(g(z), zg'(z), z^2g''(z), z^3g'''(z); z\right) : z \in U\right\}. \tag{4.8}$$

From (3.15), we deduce that the admissible condition for $\phi \in \Phi'_{\Theta}[\Omega, q]$ in Definition 4.1 is equivalent to the admissible condition for ψ as given in Definition 2.5. Hence by using the conditions in (4.5) and using Lemma 2.7, we have

$$q(z) \prec g(z), \tag{4.9}$$

or, equivalently,

$$q(z) \prec \Theta_{p,q,s}\left[\beta_1{+}1\right] f(z) \quad (z \in U). \tag{4.10}$$

This completes the proof of Theorem 4.2. □

If $\Omega \neq \mathbb{C}$ is a simply connected domain and $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω, then the class $\Phi'_{\Theta}[h(U), q]$ is written simply as $\Phi'_{\Theta}[h, q]$. With proceedings similar as in the preceding section, the following result is an immediate consequence of Theorem 4.2.

Theorem 4.3. *Let $\phi \in \Phi'_{\Theta}[h, q]$. Also, let the function h be analytic in U. If the functions $f \in \mathcal{A}(p)$ and $\Theta_{p,q,s}\left[\beta_1{+}1\right] f(z) \in Q_0$ satisfy the conditions in* (4.5) *and*

$$\phi\left(\Theta_{p,q,s}\left[\beta_1{+}1\right] f(z), \Theta_{p,q,s}\left[\beta_1\right] f(z), \Theta_{p,q,s}\left[\beta_1{-}1\right] f(z), \Theta_{p,q,s}\left[\beta_1{-}2\right] f(z); z\right),$$

is univalent in U. Then

$$h(z) \prec \phi\left(\Theta_{p,q,s}\left[\beta_1{+}1\right] f(z), \Theta_{p,q,s}\left[\beta_1\right] f(z), \Theta_{p,q,s}\left[\beta_1{-}1\right] f(z), \Theta_{p,q,s}\left[\beta_1{-}2\right] f(z); z\right), \tag{4.11}$$

implies

$$q(z) \prec \Theta_{p,q,s}\left[\beta_1{+}1\right] f(z) \quad (z \in U). \tag{4.12}$$

The following theorem proves the existence of the best subordinant of (4.11) for a suitable chosen ϕ.

Theorem 4.4. *Let the function h be analytic in U and let $\phi : \mathbb{C}^4 \times \overline{U} \longrightarrow \mathbb{C}$ and ψ be given by* (3.15). *Suppose that the differential equation*

$$\psi\left(q(z), zq'(z), z^2q''(z), z^3q'''(z); z\right) = h(z),$$

has a solution $q(z) \in Q_0$. If the functions $f \in A(p)$ and $\Theta_{p,q,s}\left[\beta_1{+}1\right] f(z) \in Q_0$ satisfy the condition (4.5) *and*

$$\phi\left(\Theta_{p,q,s}\left[\beta_1{+}1\right] f(z), \Theta_{p,q,s}\left[\beta_1\right] f(z), \Theta_{p,q,s}\left[\beta_1{-}1\right] f(z), \Theta_{p,q,s}\left[\beta_1{-}2\right] f(z); z\right),$$

is univalent in U, then

$$h(z) \prec \phi\left(\Theta_{p,q,s}\left[\beta_1{+}1\right] f(z), \Theta_{p,q,s}\left[\beta_1\right] f(z), \Theta_{p,q,s}\left[\beta_1{-}1\right] f(z), \Theta_{p,q,s}\left[\beta_1{-}2\right] f(z); z\right), \tag{4.13}$$

implies

$$q(z) \prec \Theta_{p,q,s}\left[\beta_1{+}1\right] f(z) \quad (z \in U). \tag{4.14}$$

and q is the best subordinant.

Proof. The proof of Theorem 4.4 is similar to that of Theorem 3.6 and it is being omitted here. □

Next, we introduce a new admissible class, $\widetilde{\Phi}'_{\Theta}[\Omega, q]$, as follows:

Definition 4.5. Let Ω be a set in $\mathbb{C}$, $q \in H_p$ with $q'(z) \neq 0$ and $m \in \mathbb{N}\backslash\{1\}$. The class of admissible functions $\widetilde{\Phi}'_{\Theta}[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times \overline{U} \to \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi\left(a,b,c,d;\xi\right)\in\Omega \tag{4.15}$$

whenever

$$a=q(z),\quad b=\frac{zq'(z)+m\frac{\alpha_1-pA_1-1}{A_1}q(z)}{m\frac{\alpha_1-1}{A_1}}, \tag{4.16}$$

$$Re\left\{\frac{\alpha_1(\alpha_1-1)c-(\alpha_1-pA_1)(\alpha_1-pA_1-1)a}{A_1[(\alpha_1-1)b-(\alpha_1-pA_1-1)a]}-\frac{2(\alpha_1-pA_1)-1}{A_1}\right\}\le\frac{1}{m}Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}, \tag{4.17}$$

and

$$Re\left\{\frac{\alpha_1(\alpha_1-1)(\alpha_1+1)d+(\alpha_1-pA_1)(\alpha_1-pA_1-1)(3(\alpha_1-(p-1)A_1)-(\alpha_1-pA_1+1))a-3\alpha_1(\alpha_1-1)(\alpha_1-(p-1)A_1)c}{A_1[(\alpha_1-1)b-(\alpha_1-pA_1-1)a]}\right.$$
$$\left.+\frac{3(\alpha_1-pA_1)(\alpha_1-(p-1)A_1)+(A_1-1)(3(\alpha_1-pA_1)+2A_1-1)}{A_1^2}\right\}\le\frac{1}{m^2}Re\left\{\frac{z^2q'''(z)}{q'(z)}\right\}, \tag{4.18}$$

where $m\ge 2$, $\xi\in\partial U$ and $z\in U$.

Theorem 4.6. *Let $\phi\in\widetilde{\Phi}'_{\Theta}[\Omega,q]$. If the functions $f\in\mathcal{A}(p)$ and $\Theta_{p,q,s}[\alpha_1-1]f(z)\in Q_0$ satisfy the following conditions:*

$$Re\left\{\frac{zq''(z)}{q'(z)}\right\}\ge 0,$$
$$\left|(\alpha_1-1)\,\Theta_{p,q,s}[\alpha_1+1]f(z)-(\alpha_1-pA_1-1)\,\Theta_{p,q,s}[\alpha_1]f(z)\right|\le m\left|A_1q'(z)\right|. \tag{4.19}$$

and

$$\phi\left(\Theta_{p,q,s}[\alpha_1-1]f(z),\Theta_{p,q,s}[\alpha_1]f(z),\Theta_{p,q,s}[\alpha_1+1]f(z),\Theta_{p,q,s}[\alpha_1+2]f(z);z\right),$$

is univalent in U. Then

$$\Omega\subset\left\{\phi\left(\Theta_{p,q,s}[\alpha_1-1]f(z),\Theta_{p,q,s}[\alpha_1]f(z),\Theta_{p,q,s}[\alpha_1+1]f(z),\Theta_{p,q,s}[\alpha_1+2]f(z);z\right):z\in U\right\}, \tag{4.20}$$

implies

$$q(z)\prec\Theta_{p,q,s}[\alpha_1-1]f(z)\quad(z\in U). \tag{4.21}$$

Proof. Let the function $g(z)$ be defined by (3.38) and ψ be defined by (3.45). Since $\phi\in\widetilde{\Phi}'_{\Theta}[\Omega,q]$, (3.46) and (4.20) yield

$$\Omega\subset\left\{\psi\left(g(z),zg'(z),z^2g''(z),z^3g'''(z);z\right):z\in U\right\}. \tag{4.22}$$

From (3.45), we deduce that the admissible condition for $\phi\in\widetilde{\Phi}'_{\Theta}[\Omega,q]$ in Definition 4.5 is equivalent to the admissible condition for ψ as given in Definition 2.5. Hence by using the conditions in (4.19) and using Lemma 2.7, we have

$$q(z)\prec g(z), \tag{4.23}$$

or, equivalently,

$$q(z)\prec\Theta_{p,q,s}[\alpha_1-1]f(z)\quad(z\in U), \tag{4.24}$$

this completes the proof of Theorem 4.6. □

If $\Omega\ne\mathbb{C}$ is a simply connected domain and $\Omega=h(U)$ for some conformal mapping $h(z)$ of U onto Ω, then the class $\widetilde{\Phi}'_{\Theta}[h(U),q]$ is written simply as $\widetilde{\Phi}'_{\Theta}[h,q]$. With proceedings similar as in the preceding section, the following result is an immediate consequence of Theorem 4.6.

Theorem 4.7. *Let $\phi\in\widetilde{\Phi}'_{\Theta}[h,q]$. Also, let the function h be analytic in U. If the functions $f\in\mathcal{A}(p)$ and $\Theta_{p,q,s}[\alpha_1-1]f(z)\in Q_0$ satisfy the conditions in (4.19) and*

$$\phi\left(\Theta_{p,q,s}\left[\alpha_1-1\right]f\left(z\right),\Theta_{p,q,s}\left[\alpha_1\right]f\left(z\right),\Theta_{p,q,s}\left[\alpha_1+1\right]f\left(z\right),\Theta_{p,q,s}\left[\alpha_1+2\right]f\left(z\right);z\right),$$

is univalent in U. Then

$$h(z)\prec\phi\left(\Theta_{p,q,s}\left[\alpha_1-1\right]f\left(z\right),\Theta_{p,q,s}\left[\alpha_1\right]f\left(z\right),\Theta_{p,q,s}\left[\alpha_1+1\right]f\left(z\right),\Theta_{p,q,s}\left[\alpha_1+2\right]f\left(z\right);z\right), \tag{4.25}$$

implies

$$q(z)\prec\Theta_{p,q,s}\left[\alpha_1-1\right]f\left(z\right)\ \ (z\in U). \tag{4.26}$$

The following theorem proves the existence of the best subordinant of (4.25) *for a suitable chosen ϕ.*

Theorem 4.8. *Let the function h be analytic in U and let $\phi:\mathbb{C}^4\times\overline{U}\longrightarrow\mathbb{C}$ and ψ be given by* (3.45). *Suppose that the differential equation*

$$\psi\left(q(z),zq'(z),z^2q''(z),z^3q'''(z);z\right)=h(z)$$

has a solution $q(z)\in Q_0$. If the functions $f\in A\left(p\right)$ and $\Theta_{p,q,s}\left[\alpha_1-1\right]f\left(z\right)\in Q_0$ satisfy the conditions in (4.19) *and*

$$\phi\left(\Theta_{p,q,s}\left[\alpha_1-1\right]f\left(z\right),\Theta_{p,q,s}\left[\alpha_1\right]f\left(z\right),\Theta_{p,q,s}\left[\alpha_1+1\right]f\left(z\right),\Theta_{p,q,s}\left[\alpha_1+2\right]f\left(z\right);z\right),$$

is univalent in U, then

$$h(z)\prec\phi\left(\Theta_{p,q,s}\left[\alpha_1-1\right]f\left(z\right),\Theta_{p,q,s}\left[\alpha_1\right]f\left(z\right),\Theta_{p,q,s}\left[\alpha_1+1\right]f\left(z\right),\Theta_{p,q,s}\left[\alpha_1+2\right]f\left(z\right);z\right), \tag{4.27}$$

implies

$$q(z)\prec\Theta_{p,q,s}\left[\alpha_1-1\right]f\left(z\right)\ \ (z\in U). \tag{4.28}$$

and q is the best subordinant.

Proof. The proof of Theorem 4.8 is similar to that of Theorem 3.12 and it is being omitted. □

5. Sandwich-type results

In this section, two sandwich-type results are introduced. By combining Theorems 3.4 and 4.3, we obtain the following sandwich-type result:

Theorem 5.1. *Let the functions h_1 and q_1 be analytic functions in U. Also let the function h_2 be univalent in U, $q_2\in Q_0$ with $q_1(0)=q_2(0)=0$ and $\phi\in\Phi_\Theta[h_2,q_2]\cap\Phi'_\Theta[h_1,q_1]$. If the function $f\in\mathcal{A}(p)$ and $\Theta_{p,q,s}\left[\beta_1+1\right]f\left(z\right)\in Q_0\cap H_p$ and*

$$\phi\left(\Theta_{p,q,s}\left[\beta_1+1\right]f\left(z\right),\Theta_{p,q,s}\left[\beta_1\right]f\left(z\right),\Theta_{p,q,s}\left[\beta_1-1\right]f\left(z\right),\Theta_{p,q,s}\left[\beta_1-2\right]f\left(z\right);z\right),$$

is univalent in U, and the conditions in (3.5) *and* (4.5) *are satisfied, then*

$$h_1(z)\prec\phi\left(\Theta_{p,q,s}\left[\beta_1+1\right]f\left(z\right),\Theta_{p,q,s}\left[\beta_1\right]f\left(z\right),\Theta_{p,q,s}\left[\beta_1-1\right]f\left(z\right),\Theta_{p,q,s}\left[\beta_1-2\right]f\left(z\right);z\right)\prec h_2(z), \tag{5.1}$$

implies that

$$h_1(z)\prec\Theta_{p,q,s}\left[\beta_1+1\right]f\left(z\right)\prec h_2(z). \tag{5.2}$$

Similarly, combining Theorems 3.10 and 4.7, we obtain another sandwich-type result as follows:

Theorem 5.2. *Let the functions $\widetilde{h}_1$ and $\widetilde{q}_1$ be analytic functions in U. Also let the function $\widetilde{h}_2$ be univalent in U, $\widetilde{q}_2 \in Q_0$ with $\widetilde{q}_1(0) = \widetilde{q}_2(0) = 0$ and $\phi \in \widetilde{\Phi}_\Theta[\widetilde{h}_2, \widetilde{q}_2] \cap \widetilde{\Phi}'_\Theta[\widetilde{h}_1, \widetilde{q}_1]$. If the function $f \in \mathcal{A}(p)$ and $\Theta_{p,q,s}[\alpha_1-1] f(z) \in Q_0 \cap H_p$ and*

$$\phi\left(\Theta_{p,q,s}[\alpha_1-1] f(z), \Theta_{p,q,s}[\alpha_1] f(z), \Theta_{p,q,s}[\alpha_1+1] f(z), \Theta_{p,q,s}[\alpha_1+2] f(z); z\right),$$

is univalent in U, and the conditions (3.35) and (4.19) are satisfied, then

$$\widetilde{h}_1(z) \prec \phi\left(\Theta_{p,q,s}[\alpha_1-1] f(z), \Theta_{p,q,s}[\alpha_1] f(z), \Theta_{p,q,s}[\alpha_1+1] f(z), \Theta_{p,q,s}[\alpha_1+2] f(z); z\right) \prec \widetilde{h}_2(z), \tag{5.3}$$

implies that

$$\widetilde{h}_1(z) \prec \Theta_{p,q,s}[\alpha_1-1] f(z) \prec \widetilde{h}_2(z). \tag{5.4}$$

References

[1] J. A. Antonino, S. S. Miller, *Third-order differential inequalities and subordinations in the complex plane*, Complex Var. Elliptic Equ., **56** (2011), 439–454. 2, 2.1, 2.3, 2.4, 2.6, 2.3

[2] M. K. Aouf, J. Dziok, *Certain class of analytic functions associated with the Wright generalized hypergeometric function*, J. Math. Appl., **30** (2008), 23–32. 1

[3] T. Bulboacă, *Differential subordinations and superordinations*, House of Science Book Publ. Cluj-Napoca, (2005). 1

[4] N. E. Cho, O. S. Kwon, H. M. Srivastava, *Inclusion and argument properties for certain subclass of multivalent functions associated with a family of linear operator*, J. Math. Anal. Appl., **292** (2004), 470–483. 1

[5] J. H. Choi, M. Saigo, H. M. Srivastava, *Some inclusion properties of a certain family of integral operators*, J. Math. Anal. Appl., **276** (2002), 432–445. 1, 2

[6] J. Dziok, R. K. Raina, *Families of analytic functions associated with the Wright generalized hypergeometric function*, Demonstratio Math., **37** (2004), 533–542. 1

[7] J. Dziok, H. M. Srivastava, *Classes of analytic functions associated with the generalized hypergeometric function*, Appl. Math. Comput., **103** (1999), 1–13. 1, 1

[8] A. Erdelyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, *Higher transcendental functions*, Vol. III. McGraw-Hill, New York (1954); Reprinted: Krieger, Melbourne-Florida, (1981). 1

[9] C. Fox, *The asymptotic expansion of generalized hypergeometric functions*, Proc. London Math. Soc., **27** (1928), 389–400. 1, 2

[10] R. M. Goel, N. S. Sohi, *A new criterion for p-valent functions*, Proc. Amer. Math. Soc., **78** (1980), 353–357. 1

[11] R. Gorenflo, A. A. Kilbas, F. Mainardi, S. V. Rogosin, *Mittag-Leffler functions, related topics and applications*, Springer, Heidelberg, (2014). 1

[12] A. A. Kilbas, M. Saigo, J. J. Trujillo, *On the generalized Wright function*, Fract. Calc. Appl. Anal., **5** (2002), 437–460. 1, 1

[13] V. S. Kiryakova, *Generalized fractional calculus and applications*, Pitman Research Notes in Mathematics Series, 301, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, (1994). 1

[14] V. Kumar, S. L. Shukla, *Multivalent functions defined by Ruscheweyh derivatives*, I, II. Indian J. Pure Appl. Math., **15** (1984), 1216–1227. 1

[15] O. I. Marichev, *Handbook of integral transforms and higher transcendental functions*, Theory and Algorithmic Tables. Ellis Horwood Ltd., Chichester; John Wiley & Sons, Inc., New York, (1983). 1

[16] S. S. Miller, P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J., **28** (1981), 157–172. 1

[17] S. S. Miller, P. T. Mocanu, *Differenatial subordinations: theory and applications*, Series on Monographs and Textbooks in Pure and Appl. Math. No. 255 Marcel Dekker, Inc., New York, (2000). 1, 2

[18] S. S. Miller, P. T. Mocanu, *Subordinants of differential superordinations*, Complex Var. Theory Appl., **48** (2003), 815–826. 2

[19] K. I. Noor, M. A. Noor, *On integral operators*, J. Math. Anal. Appl., **238** (1999), 341–352. 1

[20] J. Patel, A. K. Mishra, *On certain subclasses of multivalent functions associated with an extended fractional differintegral operator*, J. Math. Anal. Appl., **332** (2007), 109–122. 1

[21] H. Saitoh, *A linear operator and its applications of first order differential subordinations*, Math. Japon., **44** (1996), 31–38. 1

[22] N. Sarkar, P. Goswami, J. Dziok, J. Sokol, *Subordination for multivalent analytic functions associated with Wright generalized hypergeometric function*, Tamkang J. Math., **44** (2013), 61–71. 1

[23] H. M. Srivastava, M. K. Aouf, *A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients*, I. II. J. Math. Anal. Appl., **171** (1992), 1–13, **192** (1995), 673–688. 1

[24] H. M. Srivastava, P. W. Karlsson, *Multiple Gaussian hypergeometric series*, Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York, (1985). 1

[25] H. Tang, E. Deniz, *Third-order differential subordination results for analytic functions involving the generalized Bessel functions*, Acta Math. Sci. Ser. B Engl. Ed., **34** (2014), 1707–1719. 2

[26] H. Tang, H. M. Srivastava, E. Deniz, S.-H. Li, *Third-order differential superordination involving the generalized Bessel functions*, Bull. Malays. Math. Sci. Soc., **38** (2015), 1669–1688. 2, 2.3

[27] H. Tang, H. M. Srivastava, S.-H. Li, L.-N. Ma, *Third-order differential subordination and superordination results for meromorphically multivalent functions associated with the Liu-Srivastava operator*, Abstr. Appl. Anal., **2014** (2014), 11 pages. 2, 2.2, 2.5, 2.7, 2.3

[28] E. M. Wright, *On the coefficients of power series having exponential singularities*, J. London Math. Soc., **8** (1933), 71–79. 1

[29] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, J. London Math. Soc., **10** (1935), 286–293. 1

[30] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, Proc. London Math. Soc., **46** (1940), 389–408. 1