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Burkholder–Gundy–Davis' inequalities on weighted Lorentz martingale spaces

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Abstract

In this paper, we stablish Burkholder–Gundy–Davis' inequalities for weighted Lorentz martingale spaces $\Lambda_p(\varphi)$.

Keywords: Burkholder–Gundy–Davis' inequalities, weighted Lorentz martingale spaces, rearrangement.
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1. Introduction

It is well–known that Burkholder–Gundy–Davis' inequalities are of the fundamental inequalities in classicall martingale spaces. Burkholder and Gundy [1] proved the inequality named after them, which states that the L_p norms of the maximal function and the square function of a one parameter martingale are equivalent for $1 < p < \infty$. In 2015 the Burkholder–Gundy–Davis' inequality was proved on Lorentz martingale spaces by Ren and Guo [5]. In this paper, by using the ideas in [2] and by means of rearrangement technique we obtain a $\Lambda_p(\varphi)$–version of Burkholder–Gundy–Davis' martingale inequalities on weighted Lorentz martingale spaces.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. A filtration $(\mathcal{F}_n)_{n\in\mathbf{N}}$ is a non-decreasing sequence of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\cup_{n\in\mathbf{N}}\mathcal{F}_n)$. We denote by E the expectation operator with respect to $\mathcal{F}$.

For a martingale $f = (f_n, n \in \mathbf{N})$ relative to $(\mathcal{F}_n)_{n\geq 0}$, denote the martingale differences by $d_nf := f_n - f_{n-1}$ with convention $d_0f = 0$. The maximal function of a martingale $f = (f_n, n \in \mathbf{N})$ is denoted by

$$f_n^* := \sup_{m\leq n} |f_m|, \qquad f^* := \sup_{m\in\mathbf{N}} |f_m|.$$

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The square function of f is defined by

$$S_m(f) := \left(\sum_{n\leq m} \mid d_n f \mid^2\right)^{1/2}, \quad S(f) := \left(\sum_{n\in\mathbf{N}} \mid d_n f \mid^2\right)^{1/2},$$

Let us recall briefly the construction of weighted Lorentz spaces. For measurable function f, we define a distribution function $m(s,f)$ by setting $m(s,f) = P(\{w \in \Omega : |f(w)| > s\})$. The function

$$\widetilde{(f)}(t) = \inf\{s > 0 : m(s,f) \leq t\}, \quad t \geq 0,$$

is called the decreasing rearrangement of f.

Let $\varphi > 0$ be a non-negative and local integrable function on $[0,\infty)$. The classical weighted Lorentz spaces $\Lambda_p(\varphi)$ is defined to be the collection of all measurable functions f for which the quantity

$$\|f\|_{\Lambda_p(\varphi)} := \begin{cases} \left(\int_0^\infty \left(\widetilde{(f)}(t)\varphi(t)\right)^p \frac{dt}{t}\right)^{\frac{1}{p}} & (0 < p < \infty), \\ \sup_t \widetilde{(f)}(t)\varphi(t) & (p = \infty) \end{cases}$$

is finite. Recall that for $0 < p \leq \infty$, $\|.\|_{\Lambda_p(\varphi)}$ is only a quasi-norm.

For $0 < p \leq \infty$, weighted Lorentz-Hardy martingale spaces are defined by

$$\Lambda_p^*(\varphi) = \left\{f = (f_n)_{n\in\mathbf{N}} : \|f\|_{\Lambda_p^*(\varphi)} := \|f^*\|_{\Lambda_p(\varphi)} < \infty\right\},$$
$$\Lambda_p^S(\varphi) = \left\{f = (f_n)_{n\in\mathbf{N}} : \|f\|_{\Lambda_p^S(\varphi)} := \|S(f)\|_{\Lambda_p(\varphi)} < \infty\right\}.$$

Note that if $\varphi(t) = t^{\frac{1}{q}}$, then $\Lambda_p(\varphi) = L_{q,p}$, $\Lambda_p^*(\varphi) = H_{q,p}^*$ and $\Lambda_p^S(\varphi) = H_{q,p}^S$. In particular, if $\varphi(t) = t^{\frac{1}{p}}$, then $\Lambda_p(\varphi) = L_p$, $\Lambda_p^*(\varphi) = H_p^*$ and $\Lambda_p^S(\varphi) = H_p^S$.

Let a and b be real numbers such that $a < b$. Following Persson's convention [4], we adopt the following notations. The notation $\varphi(t) \in Q[a,b]$ means that $\varphi(t)t^{-a}$ is non–decreasing and $\varphi(t)t^{-b}$ is non-increasing for all $t > 0$. Moreover, we say that $\varphi(t) \in Q(a,b)$, wherever $\varphi(t) \in Q[a+\epsilon, b-\epsilon]$ for some $\epsilon > 0$. By $\varphi(t) \in Q(a,-)$ (or $\varphi(t) \in Q(-,b)$) we mean that $\varphi(t) \in Q(a,c)$ (or $\varphi(t) \in Q(c,b)$) for some real number c.

Our notation and terminology are standard as may be found in [6]. We use C to denote a constant, which may be different in different places.

In order to prove our main results, we collect some lemmata, which will be used in the next section.

Lemma 1.1 ([3])**.** *Let (f,g) be a pair of nonnegative measurable functions on Ω. If (f,g) satisfies the rearrangement inequality:*

$$\widetilde{(f)}(t) \leq \widetilde{(f)}(2t) + C\widetilde{(g)}(\frac{t}{2}), \quad t > 0,$$

then with the same C,

$$\widetilde{(f)}(t) \leq 2C\widetilde{(g)}(\frac{t}{2}) + \frac{C}{\log 2}\int_t^\infty \frac{\widetilde{(g)}(s)}{s}ds, \quad t > 0.$$

Lemma 1.2 ([4])**.** *Let $0 < q \leq \infty, 0 < p < \infty$ and $\psi(t) \in Q(-,-)$. Let $h(t)$ be a positive and non-increasing function on $(0,\infty)$. If $\varphi(t) \in Q(0,-)$, then*

$$\left(\int_0^\infty (\varphi(t))^q \left(\int_t^\infty (h(u)\psi(u))^p \frac{du}{u}\right)^{\frac{q}{p}} \frac{dt}{t}\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty (\varphi(t)h(t)\psi(t))^q \frac{dt}{t}\right)^{\frac{1}{q}}.$$

(C depends only on q and the constants involved in the definition of φ and ψ.)

2. Main results

Now we extend Burkholder–Gundy–Davis' inequality to $\Lambda_p(\varphi)$–quasinorm, $1 \leq p < \infty$.

Lemma 2.1. *Let $t > 0$. If $f = (f_n, n \in \mathbf{N})$ is a martingale then*

1. $\widetilde{(f^*)}(t) \leq \widetilde{(f^*)}(2t) + 2C\widetilde{S(f)}(\frac{t}{2})$
2. $\widetilde{S(f)}(t) \leq \widetilde{S(f)}(2t) + 4\acute{C}\widetilde{(f^*)}(\frac{t}{2})$.

where C and $\acute{C}$ are the constants in the inequalities $\|f^*\|_{L^2} \leq C\|S(f)\|_{L^2}$ and $\|S(f)\|_{L^1} \leq \acute{C}\|f^*\|_{L^1}$, respectivly [3].

Proof. To prove (1) we consider the following stopping times:

$$\nu = \inf\{n \in \mathcal{N} : S_n(f) > \widetilde{S(f)}(\frac{t}{2})\}, \quad \tau = \inf\{n \in \mathcal{N} : (f_n^*) > \widetilde{(f^*)}(2t)\}.$$

Then

$$\{\nu < \infty\} = \{S(f) > \widetilde{S(f)}(\frac{t}{2})\}, \quad S_{\nu-1}(f) \leq \widetilde{S(f)}(\frac{t}{2})$$
$$\{\tau < \infty\} = \{f^* > \widetilde{(f^*)}(2t)\}, \quad f^*_{\tau-1} \leq \widetilde{(f^*)}(2t).$$

Now consider $\{\mathcal{F}'_n\}_{n\geq 0}$ with $\mathcal{F}'_n = \mathcal{F}_{\tau+n}$ and $f^{\tau,\nu-1} = (f'_n)_{n\geq 0}$ with $f'_n = f^{\nu-1}_{\tau+n} - f^{\nu-1}_{\tau-1}$. Then $f^{\tau,\nu-1}$ is a martingale with respect to $\{\mathcal{F}'_n\}_{n\geq 0}$.Since

$$f^*_{\nu-1} - f^*_{(\nu-1)\wedge(\tau-1)} \leq (f^{\tau,\nu-1})^*,$$

then, we have

$$\begin{aligned}
&P\{(f^*) > \widetilde{f^*}(2t) + 2C\widetilde{S(f)}(\frac{t}{2})\} \\
&\quad\leq P\{\nu < \infty\} + P\{\nu = \infty, f^*_{\nu-1} > \widetilde{(f^*)}(2t) + 2C\widetilde{S(f)}(\frac{t}{2})\} \\
&\quad\leq \frac{t}{2} + P\{\tau < \nu = \infty, f^*_{\nu-1} - f^*_{(\nu-1)\wedge(\tau-1)} > 2C\widetilde{S(f)}\frac{t}{2}\} \\
&\quad\leq \frac{t}{2} + (2C\widetilde{S(f)}\frac{t}{2})^{-2}E[(f^*_{\nu-1} - f^*_{(\nu-1)\wedge(\tau-1)})^2\chi_{\tau<\nu}] \\
&\quad\leq \frac{t}{2} + \frac{1}{4}(C\widetilde{S(f)}\frac{t}{2})^{-2}E[((f^{\tau,\nu-1})^*)^2\chi_{\tau<\nu}] \\
&\quad\leq \frac{t}{2} + \frac{1}{4}(\widetilde{S(f)}\frac{t}{2})^{-2}E[S(f^{\tau,\nu-1})^2\chi_{\tau<\infty}] \quad (\text{by } \|f^*\|_{L^2} \leq C\|S(f)\|_{L^2}) \\
&\quad\leq \frac{t}{2} + \frac{1}{4}P\{\nu < \infty\} < \frac{t}{2} + \frac{t}{2} = t.
\end{aligned}$$

Consequently

$$\widetilde{(f^*)}(t) \leq \widetilde{(f^*)}(2t) + 2C\widetilde{S(f)}(\frac{t}{2}).$$

The inequality (2) can be proved in the same way. It is only need to consider the following stopping times:

$$\nu = \inf\{n \in \mathcal{N} : f^*_n > \widetilde{(f^*)}(\frac{t}{2})\}, \quad \tau = \inf\{n \in \mathcal{N} : S_n(f) > \widetilde{S(f)}(2t)\}.$$

□

The following Burkholder–Gundy–Davis' inequalities for martingale weighted Lorentz spaces follow from Lemmata 1.1, 1.2 and 2.1.

Theorem 2.2. *$\Lambda_p^*(\varphi)$ and $\Lambda_p^S(\varphi)$ are equvalent if $\varphi(t) \in Q(0,-)$ and $1 \le p < \infty$, namely,*

$$c\|f\|_{\Lambda_p^S(\varphi)} \le \|f\|_{\Lambda_p^*(\varphi)} \le C\|f\|_{\Lambda_p^S(\varphi)}, \quad 1 \le p < \infty,$$

for every martingale $f = (f_n, n \in \mathbf{N})$.

Proof. For any $t > 0$ we have

$$\begin{aligned}
\|f\|_{\Lambda_p^*(\varphi)} &= \left(\int_0^\infty \left(\widetilde{f^*}(t)\varphi(t)\right)^p \frac{dt}{t}\right)^{\frac{1}{p}} \\
&\le C\left(\int_0^\infty \left(\widetilde{S(f)}(\frac{t}{2})\varphi(t)\right)^p \frac{dt}{t}\right)^{\frac{1}{p}} \\
&+ C\left(\int_0^\infty (\varphi(t))^p \left(\int_t^\infty \frac{\widetilde{S(f)(s)}}{s} ds\right)^p \frac{dt}{t}\right)^{\frac{1}{p}}, \quad \text{(by Lemmata 1.1 and 2.1)} \\
&\le C\left(\int_0^\infty \left(\widetilde{S(f)}(t)\varphi(t)\right)^p \frac{dt}{t}\right)^{\frac{1}{p}}, \quad \text{(by Lemma 1.2)} \\
&= C\|f\|_{\Lambda_p^S(\varphi)}.
\end{aligned}$$

We can prove the other side of Burkholder–Gundy–Davis' inequality similarly. □

Remark 2.3. For $0 < q < \infty$, if we take $\varphi(t) = t^{\frac{1}{q}}$ in Theorem 2.2, then we get a $L_{q,p}$–version of Burkholder–Gundy–Davis' martingale inequality in martingale $H_{q,p}$ theory.

Corollary 2.4. *If $p = q$ and $1 \le q < \infty$ in Remark 2.3, we obtain the famous Burkholder–Gundy–Davis' inequality in classical martingale H_p theory.*

Note that the Burkholder–Gundy–Davis' inequalities are not valid for L_p when $0 < p < 1$. The reader is referred to [6, Proposition 2.16] for a counterexample.

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