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Some fixed point results for multivalued F-contraction on closed ball

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Abstract

We introduce the notion of Kannan type multivalued F-contraction on closed ball and obtain two new fixed point results for this contraction in a complete metric space. Some comparative examples are constructed to illustrate these results. Our results provide extensions as well as substantial generalizations and improvements of several well known results in the existing comparable literature.

Keywords: Metric space, fixed point, F contraction, closed ball.
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1. Introduction and preliminaries

Banach Contraction Principle states that any contraction on a complete metric space has a unique fixed point. This principle guarantees the existence and uniqueness of the solution of considerable problems arising in mathematics. Because of its importance for mathematical theory, Banach Contraction Principle has been extended and generalized in many directions (see [3, 9, 10, 11, 12]). The fixed point theory of multivalued contraction mappings using the Hausdorff metric was initiated by Nadler [20], who extended the Banach contraction principle to multivalued mappings. Since then many authors have studied various fixed point results for multivalued mappings. The theory of multivalued mappings has many applications in control theory, convex optimization, differential equations and economics. Recently, Sgroi and Vetro have extended the concept of F-contraction for multivalued mapping and they proved the following theorem in [25].

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Theorem 1.1 ([25]). *Let (X, d) be a complete metric space and $T : X \to CB(X)$. If there exists a mapping $F : \mathbb{R}^+ \to \mathbb{R}$, $\tau > 0$ and real numbers α, β, γ, δ, $L \geq 0$ such that*

$$2\tau + F(H(Tx, Ty)) \leq F(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + Ld(y, Tx)).$$

for all $x, y \in X$, with $Tx \neq Ty$, where $\alpha + \beta + \gamma + 2L = 1$ and $\gamma \neq 1$, then T has a fixed point.

From the application point of view the situation is not yet completely satisfactory because it frequently happens that a mapping T is a contraction not on the entire space X but merely on a subset Y of X. However, if Y is closed and a Picard iterative sequence $\{x_n\}$ in X converges to some x in X then by imposing a subtle restriction on the choice of x_0, one may force Picard iterative iterative sequence to stay eventually in Y. In this case, closedness of Y coupled with some suitable contractive condition establish the existence of a fixed point of T.

We recall some basic known definitions and results which will be used in the sequel. Throughout this paper, we denote $(0, \infty)$ by $\mathbb{R}^+$, $[0, \infty)$ by $\mathbb{R}_0^+$, $(-\infty, +\infty)$ by $\mathbb{R}$ and set of natural numbers by $\mathbb{N}$.

Definition 1.2 ([26]). Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be a F-contraction if there exists $\tau > 0$ such that

$$d(T(x), T(y)) > 0 \text{ implies } \tau + F(d(T(x), T(y))) \leq F(d(x, y)). \tag{1.1}$$

$\forall\ x, y \in X$.

Where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying following properties:

(F_1) : F is strictly increasing.

(F_2) : For each sequence $\{a_n\}$ of positive numbers $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$.

(F_3) : There exists $\theta \in (0, 1)$ such that $\lim_{\alpha\to 0^+} (\alpha)^\theta F(\alpha) = 0$.

We denote by Δ_F, the set of all functions satisfying the conditions $(F_1) - (F_3)$. Wardowski established the following result using F-contraction:

Theorem 1.3 ([26]). *Let (X, d) be a complete metric space and let $T : X \to X$ be a F-contraction. Then T has a unique fixed point $\upsilon \in X$ and for every $X_0 \in X$ a sequence $\{T^n(X_0)\}\ \forall\ \ n \in \mathbb{N}$ is convergent to υ.*

Definition 1.4 ([17]). Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be Kannan contraction if it satisfies the following condition:

$$d(T(x), T(y)) \leq \frac{k}{2}\left[d(x, T(x)) + d(y, T(y))\right]$$

for all $x, y \in X$ and some $k \in [0, 1[$.

Definition 1.5 ([23]). Let T be a self map defined on X and $\alpha : X \times X \to \mathbb{R}_0^+$ be a nonnegative function. We say that T is α-admissible if for all $x, y \in X$, $\alpha(x, y) \geq 1$ implies that $\alpha(T(x), T(y)) \geq 1$.

Definition 1.6 ([22]). Let $T : X \to X$ be a mapping and $\alpha, \eta : X \times X \to \mathbb{R}_0^+$ be two functions. We say that T is α-admissible mapping with respect to η if for all $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$ implies that $\alpha(T(x), T(y)) \geq \eta(T(x), T(y))$.

If $\eta(x, y) = 1$, then above definition reduces to Definition 1.5. If $\alpha(x, y) = 1$, then T is called an η-subadmissible mapping.

Hussain *et al.* in [14] introduced the following family of new functions.

Let Δ_G denotes the set of all functions $G : (\mathbb{R}_0^+)^4 \to \mathbb{R}^+$ which satisfy the property:
(G): for all t_1, t_2, t_3, $t_4 \in \mathbb{R}_0^+$, if $t_1t_2t_3t_4 = 0$ then there exists $\tau > 0$ such that $G(t_1, t_2, t_3, t_4) = \tau$.

Let (X, d) be a metric space. For $x \in X$ and $A \subseteq X$, we denote $d(x, A) = \inf\{d(x, y) : y \in A\}$. We denote by $N(X)$ the class of all nonempty subsets of X, by $CL(X)$ the class of all nonempty closed subsets of X, by $CB(X)$ the class of all nonempty closed and bounded subsets of X and by $K(X)$, the class of all compact subsets of X, Let H be the Hausdorff metric induced by the metric d on X, that is

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\right\}.$$

for every $A, B \in CB(X)$. If $T : X \to CB(X)$ is a multivalued mapping, then point $q \in X$ is said to be a fixed point of T if $q \in T(q)$.

Definition 1.7. Let (X, d) be a metric space. Let $T : X \to CB(X)$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be functions. We say that T is $(\alpha - \eta)$-continuous multivalued mapping on $(CB(X), H)$, if for a given $x \in X$ and a sequence $\{x_n\}$ with $x_n \overset{d}{\to} x$ as $n \to \infty$, $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, implies $T(x_n) \overset{H}{\to} T(x)$, that is $\lim_{n\to\infty} d(x_n, x) = 0$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, implies $\lim_{n\to\infty} H(T(x_n), T(x)) = 0$.

The following result play a vital role regarding the existence of the fixed point of the mapping satisfying a contractive condition on the closed ball.

Theorem 1.8 ([18], Theorem 5.1.4)**.** *Let (X, d) be a complete metric space, $T : X \to X$ be a mapping, $r > 0$ and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, 1)$ with*

$$d(T(x), T(y)) \leq kd(x, y),\ \textit{for all } x, y \in Y = \overline{B(x_0, r)}$$

and $d(x_0, T(x_0)) < (1 - k)r$. Then there exists a unique point x^* in $\overline{B(x_0, r)}$ such that $x^* = T(x^*)$.

2. Multivalued *F*-contraction on closed ball

In this section, we shall introduce the Kannan type multivalued F-contraction on closed ball and obtain a fixed point theorem for this contraction in complete metric spaces.

Definition 2.1. Let (X, d) be a metric space. The mapping $T : X \to CB(X)$ is called Kannan type multivalued F-contraction on closed ball if for all $x, y \in \overline{B(x_0, r)} \subseteq X$, we have

$$2\tau + F(H(T(x), T(y))) \leq F\left(\frac{k}{2}\left[d(x, T(x)) + d(y, T(y))\right]\right). \tag{2.1}$$

where $0 \leq k < 1$, $F \in \Delta_F$ and $\tau > 0$.

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a Kannan type multivaled F-contraction on closed ball $\overline{B(x_0, r)}$. Moreover,*

$$d(x_0, T(x_0)) \leq (1 - \lambda)r,\ \textit{where } \lambda = \frac{k}{2 - k}. \tag{2.2}$$

Then there exists a fixed point x^* in $\overline{B(x_0, r)}$.

Proof. Let $x_0 \in X$ be an arbitrary point and $x_1 \in X$. If $x_1 \in T(x_1)$, then x_1 is a fixed point of T and we are done. Assume that $x_1 \notin T(x_1)$, then $T(x_0) \neq T(x_1)$. Since F is continuous from the right, there exists a real number $h > 1$ such that

$$F\left(hH\left(T(x_0), T(x_1)\right)\right) \leq F\left(H\left(T(x_0), T(x_1)\right)\right) + \tau.$$

Choose a point x_1 in X such that $x_1 \in T(x_0)$. Continuing in this way, we get $x_{n+1} \in T(x_n)$, for all $n \geq 0$. First we show that $x_n \in \overline{B(x_0, r)}$ for all $n \in N$ by using mathematical induction method. From (2.2), we have

$$d(x_0, T(x_0)) \leq (1-\lambda)r < r. \tag{2.3}$$

Then there exists $x_1 \in T(x_0)$ such that $d(x_0, x_1) \leq (1-\lambda)r < r$, which shows that $x_1 \in \overline{B(x_0, r)}$. Suppose that $x_j \in \overline{B(x_0, r)}$ for some $j \in N$. From (2.1), we obtain

$$2\tau + F(H(T(x_0), T(x_1))) \leq F\left(\frac{k}{2}\left[d(x_0, T(x_0)) + d(x_1, T(x_1))\right]\right).$$

Since,

$$d(x_1, T(x_1)) \leq H(T(x_0), T(x_1)) < hH(T(x_0), T(x_1)).$$

There exists $x_2 \in T(x_1)$ such that

$$d(x_1, x_2) \leq hH(T(x_0), T(x_1)).$$

Which implies

$$F(d(x_1, x_2)) \leq F\left(hH(T(x_0), T(x_1))\right) \leq F\left(H(T(x_0), T(x_1))\right) + \tau.$$

Thus,

$$\begin{aligned} 2\tau + F(d(x_1, x_2)) &\leq 2\tau + F\left(H(T(x_0), T(x_1))\right) + \tau; \\ \text{implies } \ \tau + F(d(x_1, x_2)) &\leq F\left(\frac{k}{2}\left[d(x_0, x_1) + d(x_1, x_2)\right]\right). \end{aligned}$$

As F is strictly increasing, we have

$$\begin{aligned} d(x_1, x_2) &< \frac{k}{2}\left[d(x_0, x_1) + d(x_1, x_2)\right]; \\ \left(1 - \frac{k}{2}\right) d(x_1, x_2) &< \frac{k}{2} d(x_0, x_1); \\ d(x_1, x_2) &< \frac{k}{2-k} d(x_0, x_1). \end{aligned}$$

Thus, for $0 < \lambda = \frac{k}{2-k} < 1$ we have,

$$d(x_1, x_2) < \lambda d(x_0, x_1).$$

Repeating these steps for $x_3, x_4, \ldots, x_j$, we obtain

$$d(x_j, x_{j+1}) < \lambda^j d(x_0, x_1). \tag{2.4}$$

Now, using triangular inequality and (2.4), we have

$$\begin{aligned} d(x_0, x_{j+1}) &\leq d(x_0, x_1) + d\left(x_1, x_2\right) + \cdots + d(x_j, x_{j+1}); \\ &< d(x_0, x_1)\left[1 + \lambda + \lambda^2 + \cdots + \lambda^j\right]; \\ &\leq (1-\lambda)r\frac{(1-\lambda^{j+1})}{1-\lambda} < r. \end{aligned}$$

This implies that $x_{j+1} \in \overline{B(x_0, r)}$. Hence $x_n \in \overline{B(x_0, r)}$ for all $n \in N$.
Since,

$$d(x_n, T(x_n)) \le H(T(x_{n-1}), T(x_n)) < hH(T(x_{n-1}), T(x_n)).$$

So, there exists $x_{n+1} \in T(x_n)$ such that

$$d(x_n, x_{n+1}) \le hH(T(x_{n-1}), T(x_n)).$$

Now, for $x_n \notin T(x_n)$, condition (2.1) implies,

$$\begin{aligned}
2\tau + F(d(x_n, x_{n+1})) &\le 2\tau + F\left(H(T(x_{n-1}), T(x_n))\right) + \tau; \\
\tau + F(d(x_n, x_{n+1})) &\le F\left(\frac{k}{2}\left[d(x_{n-1}, T(x_{n-1})) + d(x_n, T(x_n))\right]\right) \\
&\le F\left(\frac{k}{2}\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right]\right) \\
&\le F\left(\frac{k}{2}\left[d(x_{n-1}, x_n) + \frac{k}{2-k} d(x_{n-1}, x_n)\right]\right) \\
&\le F\left(\frac{k}{2-k} d(x_{n-1}, x_n)\right) < F\left(d(x_{n-1}, x_n)\right).
\end{aligned}$$

Thus, we get

$$F\left(d(x_n, x_{n+1})\right) < F\left(d(x_{n-1}, x_n)\right) - \tau. \tag{2.5}$$

Again using F_1, we have

$$F\left(d(x_{n-1}, x_n)\right) < F\left(d(x_{n-1}, x_n)\right) \le F\left(d(x_{n-2}, x_{n-1})\right) - \tau.$$

From (2.5), we obtain

$$F\left(d(x_n, x_{n+1})\right) \le F\left(d(x_{n-2}, x_{n-1})\right) - 2\tau.$$

Repeating these steps, we get

$$F\left(d(x_n, x_{n+1})\right) \le F\left(d(x_0, x_1)\right) - n\tau. \tag{2.6}$$

From (2.6), we obtain $\lim_{n\to\infty} F\left(d(x_n, x_{n+1})\right) = -\infty$. Since $F \in \Delta_F$,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.7}$$

From the property (F_3) of F-contraction, there exists $\kappa \in (0, 1)$ such that

$$\lim_{n\to\infty} \left((d(x_n, x_{n+1}))^\kappa F\left(d(x_n, x_{n+1})\right)\right) = 0. \tag{2.8}$$

Following (2.6), for all $n \in \mathbb{N}$, we obtain

$$(d(x_n, x_{n+1}))^\kappa \left(F\left(d(x_n, x_{n+1})\right) - F\left(d(x_0, x_1)\right)\right) \le -\left(d(x_n, x_{n+1})\right)^\kappa n\tau \le 0. \tag{2.9}$$

Considering (2.7), (2.8) and letting $n \to \infty$, in (2.9), we have

$$\lim_{n\to\infty} \left(n\left(d(x_n, x_{n+1})\right)^\kappa\right) = 0. \tag{2.10}$$

Since (2.10) holds, there exists $n_1 \in \mathbb{N}$, such that $n\,(d(x_n, x_{n+1}))^{\kappa} \leq 1$ for all $n \geq n_1$ or,

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{\kappa}}} \text{ for all } n \geq n_1. \tag{2.11}$$

Using (2.11), we get for $m > n \geq n_1$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{m-1}, x_m); \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}); \\ &\leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}); \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\kappa}}}$ leads to $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in $\left(\overline{B(x_0, r)}, d\right)$. Since $\left(\overline{B(x_0, r)}, d\right)$ is a complete metric space, so there exists $x^* \in \overline{B(x_0, r)}$ such that $x_n \to x^*$ as $n \to \infty$. In order to prove that $x^* = T(x^*)$, there are two cases, (I) T is a continuous and (II) T is not continuous.

Case I: If T is continuous.

Then, the sequence $\{T(x_i)\}_{i=1}^{\infty}$ converges to $T(x^*)$ and, since $x_i \in T(x_{i-1})$ for all i, it follows that $x^* \in T(x^*)$. Hence x^* is a fixed point of T.

Case II: We assume that $H(T(x_n), T(x^*) > 0$, otherwise result is obvious. Using contractive condition (2.1), we obtain

$$2\tau + F\left(H(T(x_n), T(x^*))\right) \leq F\left(\frac{k}{2}\left[d\left(x_n, T(x_n)\right) + d\left(x^*, T(x^*)\right)\right]\right).$$

Since,

$$d(x_{n+1}, T(x^*)) \leq H(T(x_n), T(x^*)) < hH(T(x_n), T(x^*)).$$

Which implies

$$F(d(x_{n+1}, T(x^*)) \leq F\left(hH(T(x_n), T(x^*))\right) \leq F\left(H(T(x_n), T(x^*))\right) + \tau.$$

Thus,

$$\begin{aligned} 2\tau + F(d(x_{n+1}, T(x^*)) &\leq 2\tau + F\left(H(T(x_n), T(x^*))\right) + \tau; \\ \tau + F(d(x_{n+1}, T(x^*)) &\leq F\left(\frac{k}{2}\left[d(x_n, x_{n+1}) + d(x^*, T(x^*))\right]\right). \end{aligned}$$

Which implies

$$d\left(x_{n+1}, T(x^*)\right) < \frac{k}{2}\left[d\left(x_n, x_{n+1}\right) + d\left(x^*, T(x^*)\right)\right].$$

Letting $n \to \infty$ we get,

$$d(x^*, T(x^*)) < \frac{k}{2} d(x^*, T(x^*));$$

that is

$$\left(1-\frac{k}{2}\right)d(x^*,T(x^*))<0;$$

which implies

$$d(x^*,T(x^*))=0.$$

Since T is closed, thus, $x^*\in T(x^*)$ which completes the proof. □

Following example shows that the contractive condition (2.1) holds on closed ball $\overline{B(x_0,r)}$, whereas it does not hold true on the whole space.

Example 2.3. Let $X=\mathbb{R}_0^+$ and $d(x,y)=|x-y|$. Then (X,d) is a complete metric space. Define a mapping $T:X\to CB(X)$ by,

$$T(x)=\begin{cases}\left[0,\frac{x}{4}\right], & \text{if } x\in[0,1];\\ \left[x-\frac{1}{2},x-\frac{1}{4}\right], & \text{if } x\in(1,\infty).\end{cases}$$

Set $\tau=\ln(\sqrt{2})$, $k=\dfrac{3}{10}$, $x_0=\dfrac{1}{2}$, $r=\dfrac{1}{2}$, then $\overline{B(x_0,r)}=[0,1]$. If $F(\alpha)=\ln(\alpha)$, $\alpha>0$ and $\tau>0$, then

$$d(x_0,T(x_0))=\left|\frac{1}{2}-\frac{1}{8}\right|=\frac{3}{8}<(1-\lambda)r.$$

For $x,y\in\overline{B(x_0,r)}$, the inequality

$$\left|\frac{x}{4}-\frac{y}{4}\right|<\frac{k}{2}\left[\left|x-\frac{x}{4}\right|+\left|y-\frac{y}{4}\right|\right],$$

holds. Thus,

$$H(T(x),T(y))<\frac{k}{2}\left[d(x,T(x))+d(y,T(y)\right].$$

Which implies

$$2\tau+\ln\left(H(T(x),T(y))\right)\le\ln\left(\frac{k}{2}\left[d(x,T(x))+d(y,T(y)\right]\right).$$

That is

$$2\tau+F\left(H(T(x),T(y))\right)\le F\left(\frac{k}{2}\left[d(x,T(x))+d(y,T(y)\right]\right).$$

Now if $x=100$, $y=10\in(1,\infty)$, then

$$\begin{aligned}H(T(x),T(y))&=\left|x-\frac{1}{4}-y+\frac{1}{4}\right|=|x-y|.\\&\ge\frac{k}{4}=\frac{k}{2}\left[d(x,T(x))+d(y,T(y)\right].\end{aligned}$$

and consequently, contractive condition (2.1) does not hold on X. Hence, hypotheses of Theorem 2.2 hold on closed ball and $x=0$ is a fixed point of T in $\overline{B(x_0,r)}$.

Corollary 2.4. *Let (X,d) be a complete metric space and $T:X\to X$ be a Kannan type F-contraction on closed ball $\overline{B(x_0,r)}$ in complete metric space. Moreover*

$$d(x_0,T(x_0))\le(1-\lambda)r,\ \textit{where}\ \lambda=\frac{k}{2-k}.$$

Then there exists a point x^* in $\overline{B(x_0,r)}$ such that $T(x^*)=x^*$.

3. $\alpha-\eta-GF$-contraction on closed ball

In this section, we define a new contraction called Kannan type α-η-GF-multivalued contraction on closed ball and obtained a new fixed point theorem for this contraction in complete metric spaces.

Definition 3.1. Let (X,d) be a metric space. Suppose that $\alpha,\eta : X\times X\to[0,+\infty)$ are two functions. The mapping $T : X\to CB(X)$ is called a Kannan type α-η-GF- multivalued contraction on closed ball, if for all $x,y\in\overline{B(x_0,r)}\subseteq X$ with $\eta(x,T(x))\le\alpha(x,y)$ and $H(T(x),T(y))>0$, we have

$$2\tau(G)+F\left(H(T(x),T(y))\right)\le F\left(\frac{k}{2}\left[d(x,T(x))+d(y,T(y)\right]\right). \tag{3.1}$$

Where $\tau(G)=G(d(x,T(x)),d(y,T(y)),d(x,T(y)),d(y,T(x)))$, $0\le k<1, G\in\Delta_G$ and $F\in\Delta_F$.

Theorem 3.2. *Let (X,d) be a complete metric space. Let $T : X\to CB(X)$ be a multivalued Kannan type $\alpha-\eta-GF$-contraction mapping on a closed ball $\overline{B(x_0,r)}$ satisfying the following assertions:*

(1) T is an α-admissible mapping with respect to η;

(2) there exists $x_0\in X$ such that $\alpha(x_0,T(x_0))\ge\eta(x_0,T(x_0))$;

(3) $d(x_0,T(x_0))\le(1-\lambda)r$, where $\lambda=\frac{k}{2-k}$.

Then there exists a fixed point x^* of T in $\overline{B(x_0,r)}$.

Proof. Let $x_0\in X$ be an arbitrary point such that $\alpha(x_0,T(x_0))\ge\eta(x_0,T(x_0))$. Since T is an α-admissible mapping with respect to η then there exists $x_1\in T(x_0)$ such that

$$\alpha(x_0,x_1)=\alpha(x_0,T(x_0))\ge\eta(x_0,T(x_0))=\eta(x_0,x_1). \tag{3.2}$$

Continuing in such a manner, we can define a sequence $\{x_n\}\subset X$ such that $x_n\notin T(x_n)$, $x_{n+1}\in T(x_n)$, and

$$\eta(x_{n-1},x_n)=\eta(x_{n-1},T(x_{n-1}))\le\alpha(x_{n-1},T(x_{n-1}))=\alpha(x_{n-1},x_n). \tag{3.3}$$

If $x_1\in T(x_1)$, then x_1 is a fixed point of T. So, we assume that $x_0\ne x_1$, then $T(x_0)\ne T(x_1)$. Since F is continuous from the right, there exists a real number $h>1$ such that

$$F\left(hH\left(T(x_0),T(x_1)\right)\right)\le F\left(H\left(T(x_0),T(x_1)\right)\right)+\tau(G).$$

If there exists $n\in\mathbb{N}$ such that $d(x_n,T(x_n))=0$, then x_n is a fixed point of T, so we are done. We assume that $d(x_n,T(x_n))>0$, $\forall n\in\mathbb{N}$. First we show that $x_n\in\overline{B(x_0,r)}$ for all $n\in N$. From hypotheses (3) we obtain,

$$d(x_0,T(x_0))\le(1-\lambda)r<r. \tag{3.4}$$

Then there exists $x_1\in T(x_0)$ such that $d(x_0,x_1)\le(1-\lambda)r<r$, which shows that $x_1\in\overline{B(x_0,r)}$. Suppose that $x_j\in\overline{B(x_0,r)}$ for some $j\in N$. From (3.1), we obtain

$$2\tau(G)+F(H(T(x_0),T(x_1)))\le F\left(\frac{k}{2}\left[d(x_0,T(x_0))+d(x_1,T(x_1))\right]\right).$$

Since,

$$d(x_1,T(x_1))\le H(T(x_0),T(x_1))<hH(T(x_0),T(x_1)).$$

There exists $x_2\in T(x_1)$ such that

$$d(x_1,x_2)\le hH(T(x_0),T(x_1)).$$

Which implies

$$F(d(x_1,x_2)) \leq F\left(hH(T(x_0),T(x_1))\right) \leq F\left(H(T(x_0),T(x_1))\right) + \tau(G).$$

Thus,

$$\begin{aligned} 2\tau(G) + F(d(x_1,x_2)) &\leq 2\tau(G) + F\left(H(T(x_0),T(x_1))\right) + \tau(G); \\ \tau(G) + F(d(x_1,x_2)) &\leq F\left(\frac{k}{2}\left[d(x_0,x_1) + d(x_1,x_2)\right]\right). \end{aligned}$$

where $\tau(G) = G(d(x_1,T(x_1)), d(x_2,T(x_2)), d(x_1,T(x_2)), 0)$ implies that

$$d(x_1,T(x_1)).d(x_2,T(x_2)).d(x_1,T(x_2)).0 = 0.$$

Thus by property (G), there exists $\tau > 0$ such that $\tau(G) = \tau$. Therefore, we get

$$\tau + F(d(x_1,x_2)) < F\left(\frac{k}{2}\left[d(x_0,x_1) + d(x_1,x_2)\right]\right).$$

As F is strictly increasing, we have

$$\begin{aligned} d(x_1,x_2) &< \frac{k}{2}\left[d(x_0,x_1) + d(x_1,x_2)\right]; \\ \left(1 - \frac{k}{2}\right) d(x_1,x_2) &< \frac{k}{2} d(x_0,x_1); \\ d(x_1,x_2) &< \frac{k}{2-k} d(x_0,x_1). \end{aligned}$$

Thus, for $0 < \lambda = \frac{k}{2-k} < 1$ we have,

$$d(x_1,x_2) < \lambda d(x_0,x_1).$$

Repeating these steps for $x_3, x_4, \ldots, x_j$, we obtain

$$d(x_j,x_{j+1}) < \lambda^j d(x_0,x_1). \tag{3.5}$$

Now, using triangular inequality and (3.5), we have

$$\begin{aligned} d(x_0,x_{j+1}) &\leq d(x_0,x_1) + d\left(x_1,x_2\right) + \cdots + d(x_j,x_{j+1}); \\ &< d(x_0,x_1)\left[1 + \lambda + \lambda^2 + \cdots + \lambda^j\right]; \\ &\leq (1-\lambda) r \frac{(1-\lambda^{j+1})}{1-\lambda} < r. \end{aligned}$$

This implies that $x_{j+1} \in \overline{B(x_0,r)}$. Hence $x_n \in \overline{B(x_0,r)}$ for all $n \in N$.

Now, following the proof of the Theorem 2.2, we obtain for $m > n \geq n_1$,

$$d(x_n,x_m) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.$$

The convergence of the series $\sum\limits_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\kappa}}}$ entails $\lim_{n,m\to\infty} d(x_n,x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in $\left(\overline{B(x_0,r)}, d\right)$. Since $\left(\overline{B(x_0,r)}, d\right)$ is a complete metric space, so there exists $x^* \in \overline{B(x_0,r)}$ such that $x_n \to x^*$ as $n \to \infty$. In order to prove that $x^* = T(x^*)$, there are two cases, (I) T is $(\alpha-\eta)$-continuous and (II) T is not $(\alpha-\eta)$-continuous.

Case I: If T is $(\alpha-\eta)$-continuous.
Since $x_n \to x^*$ as $n \to \infty$ and $\eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n)$, for all $n \in \mathbb{N}$. Therefore α-η-continuity of T implies $T(x_n) \overset{H}{\to} T(x)$, that is, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, implies $\lim_{n\to\infty} H(T(x_n), T(x)) = 0$. Hence x^* is a fixed point of T.
Case II:
We assume that $d(x_n, T(x^*)) > 0$, otherwise x^* is a fixed point of T. From contractive condition (3.1), we obtain

$$F\left(d(x_n, T(x^*))\right) \leq F\left(\frac{k}{2}\left[d\left(x_{n-1}, x_n\right) + d\left(x^*, T(x^*)\right)\right]\right) - \tau(G).$$

Where $\tau(G) = G(d(x_{n-1}, x_n), d(x^*, T(x^*)), d(x_{n-1}, T(x^*)), d(x^*, x_n))$. Since F is continuous, we can have,

$$F\left(\lim_{n\to\infty} d(x_n, T(x^*))\right) \leq F\left(\frac{k}{2}\left[\lim_{n\to\infty} d\left(x_{n-1}, x_n\right) + \lim_{n\to\infty} d\left(x^*, T(x^*)\right)\right]\right) - \lim_{n\to\infty} \tau(G).$$

Which gives,

$$d(x^*, T(x^*)) < \frac{k}{2} d(x^*, T(x^*));$$

that is

$$\left(1 - \frac{k}{2}\right) d(x^*, T(x^*)) < 0.$$

This implies $d(x^*, T(x^*)) = 0$. Consequently, x^* is a fixed point of T in $\overline{B(x_0, r)}$. □

Example 3.3. Let $X = \mathbb{R}_0^+$ and d be the usual metric on X. Define $T : X \to X$, $\alpha : X \times X \to [0, +\infty)$, $\eta : X \times X \to \mathbb{R}^+$, $G : (\mathbb{R}_0^+)^4 \to \mathbb{R}^+$ and $F : \mathbb{R}^+ \to \mathbb{R}$ by

$$T(x) = \begin{cases} \left[0, \frac{5x}{19}\right], & \text{if } x \in [0, 1], \\ \\ \left[x - \frac{2}{3}, x - \frac{1}{3}\right], & \text{if } x \in (1, \infty). \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} e^{x+y}, & \text{if } x \in [0, 1], \\ \frac{1}{3}, & \text{otherwise.} \end{cases}$$

$\eta(x, y) = \frac{1}{2}$ for all $x, y \in X$, $G(t_1, t_2, t_3, t_4) = \tau > 0$ and $F(t) = \ln(t)$ with $t > 0$. Set $k = \frac{4}{5}$ $x_0 = \frac{1}{2}$, $r = \frac{1}{2}$, then $\overline{B(x_0, r)} = [0, 1]$. Now

$$d\left(\frac{1}{2}, T\left(\frac{1}{2}\right)\right) = \left|\frac{1}{2} - \frac{5}{38}\right| < r.$$

For if $x, y \in \overline{B(x_0, r)}$, then $\alpha(x, y) = e^{x+y} \geq \frac{1}{2} = \eta(x, y)$. On the other hand, for all $x \in [0, 1]$, $T(x) \in [0, 1]$, so $\alpha(T(x), T(y)) \geq \eta(T(x), T(y))$. Moreover, for $x \neq y$,

$$H(T(x), T(y)) = \left|\frac{5x}{19} - \frac{5y}{19}\right| > 0.$$

Clearly, $\alpha(0, T(0)) \geq \eta(0, T(0))$. Hence, we have

$$H(T(x), T(y)) = \left|\frac{5x}{19} - \frac{5y}{19}\right| = \frac{5}{19}\left|x - y\right|.$$

For $x, y \in \overline{B(x_0, r)}$, the inequality

$$\frac{5}{19}|x-y| < \frac{k}{2}\left[\left|x-\frac{5x}{19}\right| + \left|y-\frac{5y}{19}\right|\right].$$

holds. Thus,

$$H(T(x),T(y)) < \frac{k}{2}\left[d(x,T(x)) + d(y,T(y))\right].$$

Consequently,

$$2\tau + \ln\left(H(T(x),T(y))\right) \leq \ln\left(\frac{k}{2}\left[d(x,T(x)) + d(y,T(y))\right]\right).$$

Which implies

$$2\tau + F\left(H(T(x),T(y))\right) \leq F\left(\frac{k}{2}\left[d(x,T(x)) + d(y,T(y))\right]\right).$$

If $x \notin \overline{B(x_0,r)}$ or $y \notin \overline{B(x_0,r)}$, then $\alpha(x,y) = \frac{1}{3} \ngeq \frac{1}{2} = \eta(x,y)$. Moreover, if $x = 100$, $y = 10 \in (1,\infty)$, then

$$\begin{aligned} H(T(x),T(y)) &= \left|x - \frac{1}{3} - y + \frac{1}{3}\right| = |x-y|. \\ &\geq \frac{k}{3} = \frac{k}{2}\left[d(x,T(x)) + d(y,T(y)\right]. \end{aligned}$$

and consequently, contractive condition (3.1) does not hold on X. Hence, hypotheses of Theorem 3.2 hold on closed ball and $x = 0$ is a fixed point of T in $\overline{B(x_0,r)}$.

Corollary 3.4. *Let (X,d) be a complete metric space. Let $T : X \to X$ be a Kannan type $\alpha - \eta - GF$-contraction mapping on a closed ball $\overline{B(x_0,r)}$ satisfying the following assertions:*

(1) T is an α-admissible mapping with respect to η;

(2) there exists $x_0 \in X$ such that $\alpha(x_0,T(x_0)) \geq \eta(x_0,T(x_0))$;

(3) $d(x_0,T(x_0)) \leq (1-\lambda)r$, where $\lambda = \frac{k}{2-k}$.

Then there exists a unique point x^* in $\overline{B(x_0,r)}$ such that $T(x^*) = x^*$.

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