Available online at www.vonneumann-publishing.com/fatma
Func. Anal.-TMA 2 (2016), 81–87
Research Article

Functional Analysis: Theory, Methods & Applications
Journal Homepage: http://vonneumann-publishing.com/fatma

Iterative reconstruction in c-fusion frame

Reza Ahmadi

Marand Technical College, University of Tabriz, Tabriz, Iran.

Communicated by R. Saadati

Abstract

Frame of subspaces were introduced in the study of the relation between a frame and its local components and it turn out that frames of subspaces behave as a generalization of frames. In this article we introduce relaxation parameter for a c-fusion frame. Also we will present a new method for obtaining c-fusion frame bounds.

Keywords: Operator, bounded operator, iterative reconstruction, fusion frame.
2010 MSC: 42C15, 46L30.

1. Introduction

Frames was first introduced by Duffine and schaeffer [9] in the context of non harmonic Fourier series. Outside of signal processing, frames did not seem to generate much interest until the ground breaking work of Daubechies, Grossman, and Meyer [8] in 1986. Since then the theory of frames began to be more widely studied. During the last 20 years the theory of frames has been growing rapidly, since several new applications have been developed. For example, besides traditional application as signal processing, image processing, data compression, and sampling theory, frames are now used to mitigate the effect of losses in pocket-based communication systems and hence to improve the robustness of data transmission, and to design high-rate constellation with full diversity in multiple-antenna code design. The fusion frame was considered by P. G. Casazza, G. Kutyniok and S. Li in connection with distributed processing and is related to the construction of global frames [6]. The frame of subspaces theory is in fact more delicate due to complicated relations between the structure of the sequence of weighted subspaces and the local frames in the subspaces and due to the extreme sensitivity with respect to changes of the weights. Since frames, in particular fusion frame are applied in fundamental science and engineering, we consider c-fusion frame for Hilbert spaces, and extend some of the known results about bounds of frames to c-fusion frame.

Email address: rahmadi@tabrizu.ac.ir (Reza Ahmadi)

Received 2016-01-08

Throughout this paper H will be a Hilbert space and $\mathbb{H}$ will be the collection of all closed subspace of H. Also, (X, μ) will be a measure space, and $v : X \to [0, +\infty)$ a measurable mapping such that $v \neq 0$ a.e. We shall denote the unit closed ball of H by H_1.

2. Preliminaries

In the following we review the definitions and some important properties of frame theory in discrete and continuous frames. We will see that the continuous frame is an interest extension of frame which provided new tools for study the exquisite operators on a complex Hilbert space.

Definition 2.1. Let $\{f_i\}_{i\in I}$ be a sequence of members of H. We say that $\{f_i\}_{i\in I}$ is a frame for H if there exist $0 < A \leq B < \infty$ such that for all $h \in H$

$$A\|h\|^2 \leq \sum_{i\in I} | < f_i, h > |^2 \leq B\|h\|^2. \tag{2.1}$$

The constants A and B are called frame bounds. If A, B can be chosen so that $A = B$, we call this frame an A-tight frame and if $A = B = 1$ it is called a parseval frame. If we only have the upper bound, we call $\{f_i\}_{i\in I}$ a Bessel sequence. If $\{f_i\}_{i\in I}$ is a Bessel sequence then the following operators are bounded,

$$T : l^2(I) \to H,\ T(c_i) = \sum_{i\in I} c_i f_i \tag{2.2}$$

$$T^* : H \to l^2(I),\ T^*(f) = \{< f, f_i >\}_{i\in I} \tag{2.3}$$

$$Sf = TT^*f = \sum_{i\in I} < f, f_i > f_i. \tag{2.4}$$

This operators are called synthesis operator, analysis operator and frame operator, respectively.

Definition 2.2. Let (X, μ) be a measure space. Let $f : X \to H$ be weakly measurable (i.e., for all $h \in H$, the mapping $x \to < f(x), h >$ is measurable). Then f is called a continuous frame or c-frame for H if there exist $0 < A \leq B < \infty$ such that for all $h \in H$

$$A\|h\|^2 \leq \int_X | < f(x), h > |^2 d\mu \leq B\|h\|^2. \tag{2.5}$$

The representation space employed in this setting is

$$L^2(X, \mu) = \{\varphi : X \to H | \varphi \text{ is measurable and } \int_X ||\varphi(x)||^2 d\mu < \infty\}.$$

The synthesis operator, analysis operator and frame operator are defined by

$$T_f : L^2(X, \mu) \to H,\ < T_f\varphi, h >= \int_X \varphi(x) < f(x), h > d\mu(x), \tag{2.6}$$

$$T_f^* : H \to L^2(X, \mu),\ (T_f^*h)(x) =< h, f(x) >, \quad x \in X, \tag{2.7}$$

$$S_f = T_f T_f^*. \tag{2.8}$$

Also by Theorem 2.5 in [13] S_f is positive, self-adjoint and invertible.

Theorem 2.3. *Let f be a continuous frame for H with a frame operator S_f and let $V : H \to K$ be a bounded and invertible operator. Then $V \circ f$ is a continuous frame for K with the frame operator VS_fV^*.*

Proof. See [13]. $\square$

Theorem 2.4. *Let K be a closed subspace of H and let $P : H \to K$ be an orthogonal projection. The the following holds:*
(i) If f is a continuous frame for H with bounds A and B, then Pf is a continuous frame for K with the bounds A and B.
(ii) If f is a continuous frame for K with a frame operator S_f , then for each $h, k \in H$,

$$< Ph, k >= \int_X < h, S_f^{-1} f(x) >< f(x), k > d\mu(x).$$

Proof. See [13]. $\square$

Definition 2.5. For a countable index set I, let $\{W_i\}_{i\in I}$ be a family of closed subspace in H, and let $\{v_i\}_{i\in I}$ be a family of weights, i.e., $v_i > 0$ for all $i \in I$. Then $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for H if there exist $0 < C \leq D < \infty$ such that for all $h \in H$

$$C\|h\|^2 \leq \sum_{i\in I} {v_i}^2 \|\pi_{W_i}(f)\|^2 \leq D\|h\|^2 \tag{2.9}$$

where π_{W_i} is the orthogonal projection onto the subspace W_i.

We call C and D the fusion frame bounds. The family $\{(W_i, v_i)\}_{i\in I}$ is called a c-tight fusion frame, if in (2.9) the constants C and D can be chosen so that $C = D$, a parseval fusion frame provided $C = D = 1$ and an orthonormal fusion basis if $H = \bigoplus_{i\in I} W_i$. If $\{(W_i, v_i)\}_{i\in I}$ possesses an upper fusion frame bound, but not necessarily a lower bound, we call it is a Bessel fusion sequence with Bessel fusion bound D. The representation space employed in this setting is

$$\left(\sum_{i\in I} \oplus W_i\right)_{l_2} = \{\{f_i\}_{i\in I} | f_i \in W_i \text{ and } \{||f_i||\}_{i\in I} \in l^2(I)\}.$$

Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame for H. The synthesis operator, analysis operator and frame operator are defined by

$$T_W : \left(\sum_{i\in I} \oplus W_i\right)_{l_2} \to H \text{ with } T_W(f) = \sum_{i\in I} v_i f_i, \tag{2.10}$$

$$T_W^* : H \to \left(\sum_{i\in I} \oplus W_i\right)_{l_2} \text{ with } T_W^*(f) = \{v_i \pi_{W_i}(f)\}_{i\in I}, \tag{2.11}$$

$$S_W(f) = T_W T_W^* = \sum_{i\in I} v_i^2 \pi_{W_i}(f). \tag{2.12}$$

By proposition 3.7 in [6], if $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for H with fusion frame bounds C and D then S_W is a positive and invertible operator on H with $CId \leq S_W \leq DId$.

3. Main Result

Definition 3.1. Let $F : X \to \mathbb{H}$ be such that for each $h \in H$, the mapping $x \mapsto \pi_{F(x)}(h)$ is measurable (i.e. is weakly measurable). We say that (F, v) is a c-fusion frame for H if there exist $0 < A \leq B < \infty$ such that for all $h \in H$

$$A\|h\|^2 \leq \int_X v^2(x) \|\pi_{F(x)}\|^2 d\mu \leq B\|h\|^2. \tag{3.1}$$

(F, v) is called a tight c-fusion frame for H if A, B can be chosen so that $A = B$, and parseval if $A = B = 1$. If we only have the upper bound, we call (F, v) is a Bessel c-fusion mapping for H.

Definition 3.2. Let $F : X \to \mathbb{H}$. Let $L^2(X, H, F)$ be the class of all measurable mapping $f : X \to H$ such that for each $x \in X$, $f(x) \in F(x)$ and

$$\int_X \|f(x)\|^2 d\mu < \infty.$$

It can be verified that $L^2(X, H, F)$ is a Hilbert space with inner product defined by

$$< f, g >= \int_X < f(x), g(x) > d\mu,$$

for $f, g \in L^2(X, H, F)$.

Remark 3.3. For brevity, we shall denote $L^2(X, H, F)$ by $L^2(X, F)$. Let (F, v) be a Bessel c-fusion mapping, $f \in L^2(X, F)$ and $h \in H$. Then

$$\begin{aligned}
\left|\int_X v(x) < f(x), h > d\mu\right| &= \left|\int_X v(x) < \pi_{F(x)}(f(x)), h > d\mu\right| \\
&= \left|\int_X v(x) < f(x), \pi_{F(x)}(h) > d\mu\right| \\
&\leq \int_X v(x)\|f(x)\|.\|\pi_{F(x)}(h)\|d\mu \\
&\leq \left(\int_X \|f(x)\|^2 d\mu\right)^{1/2} \left(\int_X v^2(x)\|\pi_{F(x)}(h)\|^2 d\mu\right)^{1/2} \\
&\leq B^{1/2}\|h\|\left(\int_X \|f(x)\|^2 d\mu\right)^{1/2}.
\end{aligned}$$

So we may define

Definition 3.4. Let (F, v) be a Bessel c-fusion mapping for H. We define the c-fusion pre-frame operator (synthesis operator) $T_F : L^2(X, F) \to H$, by

$$< T_F(f), h >= \int_X v(x) < f(x), h > d\mu, \tag{3.2}$$

where $f \in L^2(X, F)$ and $h \in H$.

By the Remark 3.3, $T_F : L^2(X, F) \to H$ is a bounded linear mapping. For each $h \in H$ and $f \in L^2(X, F)$, we have

$$\begin{aligned}
< T_F^*(h), f > =< h, T_F(f) > &= \int_X v(x) < h, f(x) > d\mu \\
&= \int_X v(x) < \pi_{F(x)}(h), f(x) > d\mu =< v\pi_F(h), f > .
\end{aligned}$$

Hence for each $h \in H$,

$$T_F^*(h) = v\pi_F(h). \tag{3.3}$$

So $T_F^* = v\pi_F$ is the adjoint of T_F and will be called c-fusion analysis operator. $S_F = T_F T_F^*$ will be called c-fusion frame operator. The reprsentation space in this setting is $L^2(X, F)$.

Proposition 3.5. *Let A be a bounded operator on a Banach space $(B, \|.\|)$ that satisfies for some positive constant $\gamma < 1$*

$$\|f - Af\| \leq \gamma \|f\|, \ \forall f \in B. \tag{3.4}$$

Then A is invertible on B and f can be recovered from Af by the following iteration algorithm. Setting $f_o = Af$ and

$$f_{n+1} = f_n + A(f - f_n), \tag{3.5}$$

for $n \geq 0$, we have $\lim\limits_{n\to\infty} f_n = f$, With the error estimate after n iterations

$$\|f - f_n\|_B \leq \gamma^{n+1}\|f\|_B. \tag{3.6}$$

Proof. By inequality (3.4) the operator norm of $Id - A$ is less than γ. This implies that A is invertible and that the inverse can be presented as a Neumann series:

$$A^{-1} = \sum_{n=0}^{\infty}(Id - A)^n,$$

and any $f \in B$ is determined by Af and the norm-convergent series

$$f = A^{-1}Af = \sum_{n=0}^{\infty}(Id - A)^n Af.$$

The reconstruction (3.5) and the error estimate (3.6) follow easily after we have shown that the n^{th}-approximation f_n as defined in (3.4) coincides with the n^{th}- partial sum $\sum\limits_{k=0}^{n}(Id - A)^k Af$. This is clear for $n = 0$, since $f_0 = Af$ by definition. Next assume that we know already that $f = \sum\limits_{k=0}^{n}(Id - A)^k Af$. Then by induction we obtain for $n + 1$

$$\begin{aligned}\sum_{k=0}^{n+1}(Id - A)^k Af &= Af + \sum_{k=1}^{n+1}(Id - A)^k Af = Af + (Id - Af)\sum_{k=1}^{n+1}(Id - A)^k Af \\ &= Af + (Id - Af)f_n = f_n + A(f - f_n).\end{aligned}$$

Now clearly $\lim\limits_{n\to\infty} f_n = f$ and since $\sum\limits_{k=n+1}^{\infty}(Id - A)^k = (Id - A)^{n+1}A^{-1}$, we obtain

$$\|f - f_n\|_B = \left\|\sum_{k=1}^{n+1}(Id - A)^k Af\right\|_B = \|(Id - A)^{n+1}A^{-1}Af\| \leq \gamma^{n+1}\|f\|_B.$$

□

Definition 3.6. Let (F, v) be a c-fusion frame for H with bounds A, B and c-fusion frame operator S_F. We define λ-quasi frame operator $\Gamma_{\lambda F}$ for (F, v) as follow

$$\Gamma_{\lambda F} = \lambda \int_X v(x)\pi_{F(x)}(v(x)\pi_{F(x)}(h))d\mu, \tag{3.7}$$

where λ is so-called relaxation parameter.

We can prove that

$$\| h - \Gamma_{\lambda gf} \| \leq \gamma(\lambda) \| h \|, \tag{3.8}$$

where $\gamma(\lambda) = \max\{1 - \lambda A, 1 - \lambda B\}$.

Definition 3.7. Let $F : X \longrightarrow \mathbb{H}$ and $G : X \longrightarrow \mathbb{H}$ are such that for $h \in H$, the mapping $x \longrightarrow \pi_{F(x)}$ and $x \longrightarrow \pi_{G(x)}$ are measurable (i.e. weakly measurable). We define an approximation operator $A_{FG} : H \longrightarrow H$ with respect to (F, v) and (G, v) as follows

$$A_{FG} = \int_X v(x)\pi_{F(x)}(v(x)\pi_{G(x)}(h))d\mu.$$

Theorem 3.8. *Let F and G at the Definition 3.7 apply. There exist constants C_1, $C_2 > 0$, $0 \leq \gamma < 1$ such that for each $h \in H$, and $\varphi \in L^2(X, F)$ we have*
(i) $\int_X v^2(x)\|\pi_{F(x)}(h)\|^2 d\mu \leq C_1\|h\|^2$,

(ii) $\| \int_X v(x)\pi_{G(x)}(\varphi(x))d\mu \|^2 \leq C_2\|\varphi\|_2^2$,

(iii) $\|h - \int_X v(x)\pi_{F(x)}(v(x)\pi_{G(x)}(h))d\mu\| \leq \gamma\|h\|$.

Then (F, v) is a c-fusion frame with bounds $\frac{(1-\gamma)^2}{C_2}$ and C_1, also (G, v) is a c-fusion frame with bounds $\frac{(1-\gamma)^2}{C_1}$ and and C_2.

Proof. Let A_{FG} be defined as in Definition 3.7, then A_{FG} is bounded operator on H because for each $h \in H$, assuming $\varphi(x) = v(x)\pi_{F(x)}$ then (i) results in $\varphi \in L^2(X, F)$ and by (i) and (ii) we have

$$\|A_{FG}\|^2 = \| \int_X v(x)\pi_{F(x)}(v(x)\pi_{G(x)}(h))d\mu\|^2 \leq C_2 \int_X v^2(x)\|\pi_{F(x)}\|^2 d\mu \leq C_1C_2\|h\|^2.$$

By Proposition 3.5 A_{FG} is invertible with $A_{FG}^{-1} = \sum_{n=0}^{\infty}(Id - A_{FG})^n$ and $\|A_{FG}^{-1}\| \leq (1 - \gamma)^{-1}$. Now by (i) and (ii) we have

$$\begin{aligned}\|h\|^2 = \|A_{FG}^{-1}A_{FG}(h)\|^2 \leq (1-\gamma)^{-2}\|A_{FG}(h)\|^2 &= (1-\gamma)^{-2}\left\|\int_X v(x)\pi_{F(x)}(v(x)\pi_{G(x)}(h))d\mu\right\|^2 \\ &\leq C_2(1-\gamma)^{-2}\int_X v^2(x)\|\pi_{F(x)}\|^2 d\mu \leq C_1C_2(1-\gamma)^{-2}\|h\|^2.\end{aligned}$$

We conclude that (F, v) is a c-fusion frame with bounds $\frac{(1-\gamma)^2}{C_2}$ and C_1. Next we verify two inequalities which are dual to (i) and (ii),

$$\begin{aligned}\left(\int_X v^2(x)\|\pi_{G(x)}(h)\|^2 d\mu\right)^2 &= \left(\left\langle \int_X v(x)\pi_{G(x)}(h)(v(x)\pi_{G(x)}(h))d\mu, h\right\rangle\right)^2 \\ &\leq \left\|\int_X v(x)\pi_{G(x)}(h)(v(x)\pi_{G(x)}(h))d\mu\right\|^2 \|h\|^2 \leq C_2\|h\|^2 \int_X v^2(x)\|\pi_{G(x)}(h)\|^2 d\mu,\end{aligned}$$

hence

$$\int_X v^2(x)\|\pi_{G(x)}(h)\|^2 d\mu \leq C_2\|h\|^2.$$

Now let $\varphi \in L^2L(X, F)$, we have

$$\left\|\int_X v(x)\pi_{G(x)}(\varphi(x))d\mu\right\| = \sup_{\|h\|=1}\left|\left\langle h, \int_X v(x)\pi_{G(x)}(\varphi(x))d\mu\right\rangle\right|,$$

and

$$\begin{aligned}&\left|\left\langle h, \int_X v(x)\pi_{G(x)}(\varphi(x))d\mu\right\rangle\right|^2 = \left|\int_X \langle v(x)\pi_{G(x)}(h), \varphi(x)\rangle d\mu\right| \\ &\|\varphi\|^2 \int_X v^2(x)\|\pi_{G(x)}(h)\|^2 d\mu_1\|\varphi\|^2\|h\|^2.\end{aligned}$$

Now by similar argument and applying an approximation operator of the form

$$A_{GF} = \int_X v(x)\pi_{F(x)}(v(x)\pi_{G(x)}(h))d\mu,$$

we can establish (G, v) has required properties. □

References

[1] J. J. Benedetto, A. Powell, Ö. Yilmaz, *Sigma-Delta quantization and finite frames*, IEEE Trans. Acoustics, speech, sign. proc., **3** (2004), 937–940.

[2] H. Bölcskei, F. Hlawatsch, H. G. Feichtinger, *Frame-theoretic analysis of oversampled filter bank*, IEEE Trans. Sign. Proc., **46** (1998), 3256–3268.

[3] E. J. Candes, D. L. Donoho, *New tight frames of curvelets and optimal representation of objects with piecewise C^2 singularities*, Comm. Pure and Appl. Math., **57** (2004), 219–266.

[4] P. G. Casazza, J. Kovaĉević, *Equal-norm tight frames with erasures*, Frames. Adv. Comput. Math., **18** (2003), 387–430.

[5] P. G. Casazza, G. Kutyniok, *Frame of subspaces*, Contemp. Math., **345** (2004), 87–113.

[6] P. G. Casazza, G. Kutyniok, S. Li, *Fusion frames and distributed processing*, Appl. Comput. Harmon. Anal., **25** (2008), 114–132. 1, 2

[7] O. Christensen, *Introduction to frames and Riesz bases*, Applied and Numerical Harmonic Analysis. Birkhuser Boston, Inc., Boston, MA, (2003).

[8] I. Daubechies, A. Grossmann, Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys., **27** (1986), 1271–1283. 1

[9] R. J. Duffin, A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., **72** (1952), 341–366. 1

[10] M. H. Faroughi, R. Ahmadi, *Fusion integral*, Math. Nachr., **284** (2011), 681–693.

[11] J. P. Gabardo, D. Han, *Frames associated with measurable space*, Frames. Adv. Comput. Math., **18** (2003), 127–147.

[12] Gert K. Pedersen, *Analysis now*, Springer-Verlag, New York, (1989).

[13] A. Rahimi, A. Najati, Y. N. Dehgan, *Continuous frame in Hilbert space*, Methods Funct. Anal. Topology, **12** (2006), 170–182. 2, 2, 2

[14] W. Rudin, *Real and complex analysis*, Third edition, McGraw-Hill Book Co., New York, (1987).

[15] W. Rudin, *Functional analysis*, Second edition, International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, (1991).

[16] S. Sakai, *C^*-Algebras and W^*-Algebras*, Springer-Verlag Berlin Heidelberg, (1966).

[17] A. Shokrollahi, B. Hassibi, B. M. Hochwald, W. Sweldens, *Representation theory for high-rate multiple-antenna code design*, IEEE Trans. Inform. Theory, **47** (2001), 2335–2367.