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Fixed point theorems for a pair of weakly increasing self maps under Geraghty contractions in partially ordered partial b-metric spaces

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Abstract

In this paper we consider the concept of generalized Geraghty contractive condition for a pair of weakly increasing self maps in a complete partially ordered partial b-metric space. We study the existence of fixed points for such a pair of weakly increasing self maps in a complete partially ordered partial b-metric spaces controlled by generalized Geraghty contractive type condition and obtain some fixed point results of V. La Rosa *et al.* [15] in a complete partially ordered partial b-metric spaces as corollaries. Supporting example is also provided.

Keywords: Fixed point theorems, weakly increasing mappings, coupled $\alpha-$admissible, contractive mappings, partial metric, partial b-metric, ordered partial metric space, partially ordered partial b-metric space, Geraghty contraction.
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1. Introduction

Fixed point theorems usually start from Banach [7] contraction principle. But all the generalizations may not be from this principle. In 1973, Geraghty [10] introduced an extension of the contraction in which the contraction constant was replaced by a function having some specified properties. In 1989, Bakhtin [6] introduced the concept of a b-metric space as a generalization of a metric spaces. In 1993, Czerwik [9] extended many results related to the b-metric spaces. In 1994, Matthews [16] introduced the concept of partial metric space in which the self distance of any point of space may not be zero. In 1996, O'Neill [21] generalized the concept of partial metric space by admitting negative distances. Moreover, the existence of several connections between partial metrics and topological aspects of domain theory have been pointed

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by O'Neill [21]. In 2013, Shukla [26] generalized both the concepts of b-metric and partial metric space by introducing the partial b-metric spaces. Many authors recently studied the existence of fixed points of self maps in different types of metric spaces [2, 5, 12, 20, 23, 24, 26]. Xian Zhang [29] proved a common fixed point theorem for two self maps on a metric space satisfying generalized contractive type conditions. Some authors studied some fixed point theorems in b-metric spaces [14, 17, 23, 24, 26]. After that some authors started to prove α-ψ versions of certain fixed point theorems in different type metric spaces [12, 13, 22, 23]. Mustafa [19] gave a generalization of Banach contraction principle in complete ordered partial b-metric space by introducing a generalized α-ψ weakly contractive mapping. Aiman Mukheimer [17] generalized the concept of Mustafa [19] by introducing the α-φ-ψ contractive mapping in a complete ordered partial b-metric space.
In this paper we prove fixed point theorems by using generalized Geraghty contractive condition for a pair of weakly increasing self maps in a complete partially ordered partial b-metric space. We study the existence of fixed points for such a pair of weakly increasing self maps in complete partially ordered partial b-metric spaces controlled by generalized Geraghty contractive type condition and obtain some fixed point results of V. La Rosa *et al.* [15] in complete partially ordered partial b-metric spaces as corollaries. Supporting example is also provided. Shukla [26] introduced the notation of a partial b-metric space as follows.

2. Preliminaries

We first offer several basic facts used throughout this paper.

Definition 2.1 (S. Shukla [26]). Let X be a non empty set and let $s \geq 1$ be a given real number. A function $p : X \times X \to [0, \infty)$ is called a partial
b-metric if for all $x, y, z \in X$ the following conditions are satisfied.
(i) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,
(ii) $p(x, x) \leq p(x, y)$,
(iii) $p(x, y) = p(y, x)$,
(iv) $p(x, y) \leq s\{p(x, z) + p(z, y)\} - p(z, z)$.
The pair (X, p) is called a partial b-metric space. The number $s \geq 1$ is called a coefficient of (X, p).

Definition 2.2 (E. Karapinar, B. Samet [13]). Let $(X, \leq)$ be a partially ordered set and $f : X \to X$ be a mapping. We say that f is non decreasing with respect to $\leq$ if $x, y \in X$, $x \leq y \Rightarrow fx \leq fy$.

Definition 2.3 (E. Karapinar, B. Samet [13]). Let $(X, \leq)$ be a partially ordered set. A sequence $\{x_n\} \in X$ is said to be non decreasing with respect to $\leq$ if $x_n \leq x_{n+1}$, $\forall\ n \in \mathbb{N}$.

Definition 2.4 (Z. Mustafa [19]). A triple $(X, \leq, p)$ is called an ordered partial b-metric space if $(X, \leq)$ is a partially ordered set and p is a partial b-metric on X.

Definition 2.5 (M. A. Geraghty [10]). A self map $f : X \to X$ is said to be a Geraghty contraction if there exists $\beta \in \Omega$ such that $d(f(x), f(y)) \leq \beta(d(x, y))d(x, y)$ where $\Omega = \{\beta : [0, \infty) \to [0, 1)/\beta(t_n) \to 1 \Rightarrow t_n \to 0\}$.

Definition 2.6 (B. Samet *et al.* [22]). Suppose $(X, \leq, p)$ is a partially ordered partial b-metric space and $f : X \to X$ is a self map. Let $\alpha : X \times X \to [0, \infty)$. f is said to be $\alpha-$admissible if forall $x, y \in X$, $\alpha(x, y) \geq 1 \Rightarrow \alpha(fx, fy) \geq 1$.

Definition 2.7 (E. Karapinar, B. Samet [13]). An $\alpha-$admissible map T is said to be triangular $\alpha-$admissible if $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1 \Rightarrow \alpha(x, y) \geq 1$.

Lemma 2.8 (E. Karapinar, B. Samet [13]). *Let $T : X \to X$ be triangular α admissible map. Assume that there exists $x_1 \in X \ni \alpha(x_1, Tx_1) \geq 1$. Define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n, n = 0, 1, 2, \ldots$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Definition 2.9 (I. Beg, A. R. Butt [8]). Let $(X, \leq)$ be a partially ordered set and $S, T : X \to X$ be such that $Sx \leq TSx$ and $Tx \leq STx$, $\forall\ x \in X$. Then S and T are said to be weakly increasing mappings.

Definition 2.10 (J. Hassanzadeasl [11]). Let $T, S : X \to X$, and let $\alpha : X \times X \to [0, \infty)$. We say that S, T are coupled $\alpha-$admissible if $\alpha(x, y) \geq 1 \Rightarrow \alpha(Sx, Ty) \geq 1$ and $\alpha(Tx, Sy) \geq 1$ for all $x, y \in X$.

Definition 2.11 (V. La Rosa *et al.* [15]). Let $(X, \leq)$ is a partially ordered set and suppose that there exists a partial metric p such that (X, p) is a partial metric space. Let f be a self mapping on X. If there exists $\beta \in \Omega$ such that $p(f(x), f(y)) \leq \beta(M(x, y))M(x, y)$ for all $x, y \in X$ with

$$M(x, y) = \max\left\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2}[p(x, fy) + p(fx, y)]\right\},$$

then we say that f is a generalized Geraghty contraction map.

Definition 2.12 (V. La Rosa *et al.* [15]). Let $(X, \leq)$ is a partially ordered set and suppose that there exists a partial metric p such that (X, p) is a partial metric space. Let $\alpha : X \times X \to [0, \infty)$. X is called $\alpha-$regular If for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geq 1\ \forall\ n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$, then there exists a sub sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1\ \forall\ k \in \mathbb{N}$.

V. La Rosa *et al.* [15] proved the following theorems.

Theorem 2.13 (V. La Rosa *et al.* [15] Theorem 3.5). *Let $(X, \leq, p)$ be a complete partial metric space and let $\alpha : X \times X \to [0, \infty)$ be a function. Let $f : X \to X$ be a self mapping. Suppose that there exists $\beta \in \Omega$ such that $\alpha(x, fx)\alpha(y, fy)p(fx, fy) \leq \beta(M(x, y))M(x, y)$ for all $x, y \in X$, where*

$$M(x, y) = \max\left\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2}[p(x, fy) + p(fx, y)]\right\}.$$

Assume that
(i) f is α admissible,
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$,
(iii) for every sequence $\{x_n\}$ in X such that $\alpha(x_n, fx_n) \geq 1\ \ \forall\ n \in \mathbb{N} \cup \{0\}$ and $\{x_n\}$ converges to x, then $\alpha(x, fx) \geq 1$,
(iv) $\alpha(x, fx) \geq 1\ \forall\ x \in Fix(f)$,
then f has a unique fixed point x in X.

Theorem 2.14 (V. La Rosa *et al.* [15] Theorem 3.6). *Let $(X, \leq, p)$ be a complete partial metric space and let $\alpha : X \times X \to [0, \infty)$ be a function. Let $f : X \to X$ be a self mapping. Suppose that there exists $\beta \in \Omega$ such that $\alpha(x, y)p(fx, fy) \leq \beta(M(x, y))M(x, y)$ for all $x, y \in X$, where*

$$M(x, y) = \max\left\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2}[p(x, fy) + p(fx, y)]\right\}.$$

Assume that
(i) f is α admissible,
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$,
(iii) X is $\alpha-$regular and for every sequence $\{x_n\} \subset X$ such that
$\alpha(x_n, x_{n+1}) \geq 1\ \ \forall\ n \in \mathbb{N} \cup \{0\}$, we have $\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$,
(iv) $\alpha(x, y) \geq 1\ \forall\ x, y \in Fix(f)$,
then f has a unique fixed point $x \in X$.

Theorem 2.15 (V. La Rosa *et al.* [15] Theorem 4.1). *Let $(X, \leq, p)$ be a complete ordered partial metric space and let $\alpha : X \times X \to [0, \infty)$ be a function. Let $f : X \to X$ be a non-decreasing mapping. Suppose that there exists $\beta \in \Omega$ such that $p(fx, fy) \leq \beta(M(x, y))M(x, y)$ for all $x, y \in X$ with $x \leq y$, where*

$$M(x,y) = \max\left\{p(x,y), p(x,fx), p(y,fy), \frac{1}{2}[p(x,fy)+p(y,fx)]\right\}.$$

Assume also that the following conditions hold:
(i) there exists $x_0 \in X$ such that $x_0 \leq fx_0$,
(ii) X is such that, if a non-decreasing sequence $\{x_n\}$ converges to x, then there exists a sub sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \leq x \; \forall \; k \in \mathbb{N}$,
(iv) x, y are comparable whenever $x, y \in Fix(f)$,
then f has a unique fixed point $x \in X$.

3. Main results

In this section we extend the study of Theorems 2.13, 2.14 and 2.15 for partially ordered partial b-metric spaces by using by partial b-metric p of Definition 2.1 and a pair of weakly increasing self maps controlled by generalized Geraghty contraction. We begin this section with the following definition:

Definition 3.1. Suppose $(X, \leq)$ is a partially ordered set and p is a partial b-metric in the sense of Definition 2.1 with $s \geq 1$ as the coefficient of (X,p). Then we say that the triplet $(X, \leq, p)$ is a partially ordered partial b-metric space. A partially ordered partial b-metric space $(X, \leq, p)$ is said to be complete if every Cauchy sequence in X is convergent in the sense of the Definition 2.1. We observe that every ordered partial b-metric space is a partially ordered partial b-metric space, in the light of the observation made above.

Definition 3.2. Let $(X, \leq)$ is a partially ordered set and suppose that there exists a partial b-metric p such that (X,p) is a partial b-metric space with $s \geq 1$ be the coefficient. Let f be a self mapping on X. If there exists $\beta \in \Omega$ such that $sp(f(x), f(y)) \leq \beta(M(x,y))M(x,y)$ for all $x, y \in X$ where

$$M(x,y) = \max\left\{p(x,y), p(x,fx), p(y,fy), \frac{1}{2s}[p(x,fy)+p(fx,y)]\right\},$$

then we say that f is a generalized Geraghty contraction map.

Now we state the following useful lemmas, whose proofs can be found in Sastry *et al.* [24].

Lemma 3.3. *Let $(X, \leq, p)$ be a p complete partially ordered partial b-metric space with coefficient $s \geq 1$. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$. Suppose $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$. Then $\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, y) = p(x,y)$ and hence $x = y$.*

Lemma 3.4. *(i) $p(x,y) = 0 \Rightarrow x = y$;*
(ii) $\lim_{n\to\infty} p(x_n, x) = 0 \Rightarrow p(x,x) = 0$ and hence $x_n \to x$ as $n \to \infty$.

Lemma 3.5. *Let $(X, \leq, p)$ be a partially ordered partial b-metric space with coefficient $s \geq 1$. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$.*
Then
(i) $\{x_n\}$ is a Cauchy sequence $\Rightarrow \lim_{m,n\to\infty} p(x_m, x_n) = 0$;
(ii) $\{x_n\}$ is not a Cauchy sequence $\Rightarrow \exists \; \epsilon > 0$ and sequences $\{m_k\}$, $\{n_k\} \ni m_k > n_k > k \in \mathbb{N}$; $p(x_{n_k}, x_{m_k}) > \epsilon$ and $p(x_{n_k}, x_{m_k - 1}) \leq \epsilon$.

Proof. (i) Suppose $\{x_n\}$ is a Cauchy sequence then $\lim_{m,n\to\infty} p(x_m, x_n)$ exists and finite.
Therefore $0 = \lim_{n\to\infty} p(x_n, x_{n+1}) = \lim_{m,n\to\infty} p(x_m, x_n)$. Therefore $\lim_{m,n\to\infty} p(x_m, x_n) = 0$.

(ii) $\{x_n\}$ is not a Cauchy sequence $\Rightarrow \lim_{m,n\to\infty} p(x_m, x_n) \neq 0$ if it exists
$\Rightarrow \exists\ \epsilon > 0$ and for every $N \in \mathbb{N}$ and for $m, n \in \mathbb{N}$; $m, n > N \ni p(x_m, x_n) > \epsilon$,

$$\because \lim_{n\to\infty} p(x_n, x_{n+1}) = 0 \Rightarrow \exists\ M \in \mathbb{N} \ni p(x_n, x_{n+1}) < \epsilon\ \forall\ n > M.$$

Let $N_1 > M$ and n_1 be the smallest such that $m > n_1$ and $p(x_{n_1}, x_m) > \epsilon$ for at least one m. Let m_1 be the smallest such that $m_1 > n_1 > N_1 > 1$ and $p(x_{n_1}, x_{m_1}) > \epsilon$ so that $p(x_{n_1}, x_{m_1 - 1}) \leq \epsilon$. Let $N_2 > N_1$ and choose $m_2 > n_2 > N_2 > 2 \ni p(x_{n_2}, x_{m_2}) > \epsilon$ and $p(x_{n_2}, x_{m_2 - 1}) \leq \epsilon$.
Continuing this process we can get sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that $m_k > n_k > k$ and $p(x_{m_k}, x_{n_k}) > \epsilon$; $p(x_{n_k}, x_{m_k - 1}) \leq \epsilon$. □

Lemma 3.6. *Let $(X, \leq, p)$ be a partially ordered partial b-metric space with coefficient $s \geq 1$. Let $\{x_n\}$ be a sequence in $X \ni sp(x_n, y) \leq p(x, y)$ and $\{x_n\} \to x$ as $n \to \infty$, then $\{sp(x_n, y)\} \to p(x, y)$ as $n \to \infty$.*

Proof. Since $sp(x_n, y) \leq p(x, y)$, then $\limsup_{n\to\infty} sp(x_n, y) \leq p(x, y)$. On the other hand

$$\begin{aligned} p(x, y) &\leq sp(x, x_n) + sp(x_n, y) - p(x_n, x_n) \\ &\leq sp(x, x_n) + sp(x_n, y), \end{aligned}$$

$$\begin{aligned} &\Rightarrow p(x, y) \leq \liminf_{n\to\infty} sp(x_n, y), \\ &\therefore \limsup_{n\to\infty} sp(x_n, y) \leq p(x, y) \leq \liminf_{n\to\infty} sp(x_n, y), \\ &\therefore \lim_{n\to\infty} sp(x_n, y) = p(x, y). \end{aligned}$$

□

Now we state our first main result:

Theorem 3.7. *Let $(X, \leq, p)$ be a complete partially ordered partial b-metric space with $s \geq 1$ and let $\alpha : X \times X \to [0, \infty)$ be a function such that $\alpha(x, x) \geq 1\ \forall\ x \in X$. Let $S, T : X \to X$ be a pair of self maps. Suppose that there exists $\beta \in \Omega$ such that $\alpha(x, Sx)\alpha(y, Ty)sp(Sx, Ty) \leq \beta(M(x, y))M(x, y)$ for all $x, y \in X$, where*

$$M(x, y) = \max\left\{p(x, y), p(x, Sx), p(y, Ty), \frac{1}{2s}[p(x, Ty) + p(Sx, y)]\right\}. \tag{3.1}$$

Assume that
(i) S, T are weakly increasing,
(ii) S, T are coupled $\alpha-$admissible and $\alpha-$triangular admissible,
(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$,
(iv) for every sequence $\{x_n\}$ in X such that $\{x_n\}$ converges to x, then $\alpha(x, Sx) \geq 1$ and $\alpha(x, Tx) \geq 1$,
then S, T have a unique common fixed point in X.

Proof. We first prove that any fixed point of S is also a fixed point of T and conversely. Let x be a fixed point of S. Then $Sx = x$. Now

$$M(x, x) = \max\left\{p(x, x), p(Sx, x), p(Tx, x), \frac{1}{2}[p(Sx, x) + p(Tx, x)]\right\} = p(Tx, x),$$

$$\begin{aligned}\therefore p(x,Tx) &\leq sp(Sx,Tx),\\ &\leq \alpha(x,Sx)\alpha(x,Tx)sp(Sx,Tx)),\\ &\leq \beta(M(x,x))(M(x,x)),\\ &= \beta(p(x,Tx))(p(x,Tx)),\\ &= (p(x,Tx)),\end{aligned}$$

only if $\beta\ (p(x,Tx)) = 1 \Rightarrow p(x,Tx) = 0$,
$\therefore p(x,Tx) = 0$,
$\therefore$ by Lemma 3.4 (i) $Tx = x$.
Similarly if $Tx = x$ then $Sx = x$.
Further we show that if S and T have a common fixed point then it is unique. Let $Tx = Sx = x$ and $Ty = Sy = y$. To show that $x = y$. Suppose $x \neq y$. We have

$$M(x,y) = \max\left\{p(x,y), p(Sx,x), p(Ty,y), \frac{1}{2}[p(Sx,y) + p(Ty,x)]\right\} = p(x,y),$$

$$\begin{aligned}\therefore p(x,y) &\leq sp(Sx,Ty)),\\ &\leq \alpha(x,Sx)\alpha(y,Ty)sp(Sx,Ty)),\\ &\leq \alpha(x,x)\alpha(y,y)sp(x,y)),\\ &\leq \beta(p(x,y))(p(x,y)),\\ &= p(x,y),\end{aligned}$$

only if $\beta\ (p(x,y)) = 1 \Rightarrow p(x,y) = 0$,
$\therefore$ by Lemma 3.4 (i) $x = y$, a contradiction.
$\therefore x = y$.
Let $x_0 \in X$ and $x_{2n+1} = Sx_{2n}$;
$x_{2n+2} = Tx_{2n+1}$; $n = 0, 1, 2, \cdots$.
For any n suppose $x_{n+1} = x_n$.
Now $n = 2m$,
$\Rightarrow x_{2m+1} = x_{2m}$,
$\Rightarrow Sx_{2m} = x_{2m}$,
$\Rightarrow x_n$ is a fixed point of S.
For $n = 2m + 1$,
$\Rightarrow x_{2m+2} = x_{2m+1}$,
$Tx_{2m+1} = x_{2m+1}$,
$\Rightarrow x_n$ is a fixed point of T.
$\therefore$ For any n if $x_{n+1} = x_n$ then x_n is a common fixed point of T and S.
Hence for any n, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$.
Since S and T are weakly increasing,

$$x_1 = Sx_0 \leq TSx_0 = Tx_1 = x_2 \leq STx_1 = Sx_2 = x_3 \cdots .$$

$\therefore x_1 \leq x_2 \leq x_3 \leq \cdots$. Thus $\{x_n\}$ is increasing.
Let $x_0 \in X$ be such that $\alpha(x_0, Sx_0) \geq 1$ by (iii). Without loss of generality, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By using the α-admissibility of T , we have $\alpha(x_0, x_1) = \alpha(x_0, Sx_0) \geq 1 \Rightarrow \alpha(x_1, x_2) = \alpha(Sx_0, Tx_1) \geq 1$. Now, by mathematical induction, it is easy to see that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.

Let n be even and by taking $x = x_{n-1}$ and $y = x_n$ in the inequality (3.1), and observing that $p(x_{n-1}, x_n) \neq 0$ by Lemma 3.3, and $\alpha(x_{n-1}, Tx_{n-1}) \geq 1$,$\alpha(x_n, Sx_n) \geq 1$, we get

$$\begin{aligned} p(x_n, x_{n+1}) &\leq sp(x_n, x_{n+1}) \\ &\leq \alpha(x_{n-1}, Tx_{n-1})\alpha(x_n, Sx_n)sp(Tx_{n-1}, Sx_n)) \\ &\leq \beta(M(x_n, x_{n-1}))(M(x_n, x_{n-1})), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\left\{p(x_{n-1}, x_n), p(x_{n-1}, Tx_{n-1}), p(x_n, Sx_n), \frac{1}{2s}[p(x_{n-1}, Sx_n) + p(x_n, Tx_{n-1})]\right\} \\ &= \max\left\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2s}[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)]\right\} \\ &\leq \max\left\{p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2s}[sp(x_{n-1}, x_n) + sp(x_n, x_{n+1}) - p(x_n, x_n) + p(x_n, x_n)]\right\} \\ &= \max\left\{p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2}[p(x_{n-1}, x_n) + p(x_n, x_{n+1})]\right\} \\ &= \max\left\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\right\}. \end{aligned}$$

If

$$\max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_n, x_{n+1}), \tag{3.3}$$

for some $n \in \mathbb{N}$ then from (3.2) and (3.3), we have $p(x_n, x_{n+1})) \leq M(x_{n-1}, x_n) = p(x_n, x_{n+1})$, which is possible only if $\beta(p(x_n, x_{n+1})) = 1 \Rightarrow p(x_n, x_{n+1}) = 0$ a contradiction.
Thus, we have $M(x_{n-1}, x_n) = \max\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\} = p(x_{n-1}, x_n)$ Similarly, Let n be odd and by taking $x = x_{n-1}$ and $y = x_n$ in the inequality (3.1), and observing that $p(x_{n-1}, x_n) \neq 0$ by lemma 3.3, we get

$$\begin{aligned} p(x_n, x_{n+1}) &\leq sp(x_n, x_{n+1}) \\ &= sp(Sx_{n-1}, Tx_n)) \\ &\leq \alpha(x_{n-1}, Sx_{n-1})\alpha(x_n, Tx_n)sp(Tx_{n-1}, Sx_n)) \\ &< \beta(M(x_{n-1}, x_n))(M(x_{n-1}, x_n)), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{p(x_{n-1}, x_n), p(x_{n-1}, Sx_{n-1}), p(x_n, Tx_n), \frac{1}{2s}[p(x_{n-1}, Tx_n) + p(x_n, Sx_{n-1})]\right\} \\ &= \max\left\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2s}[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)]\right\} \\ &\leq \max\left\{p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2s}[sp(x_{n-1}, x_n) + sp(x_n, x_{n+1}) - p(x_n, x_n) + p(x_n, x_n)]\right\} \\ &= \max\left\{p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2}[p(x_{n-1}, x_n) + p(x_n, x_{n+1})]\right\} \\ &= \max\left\{p(x_{n-1}, x_n), p(x_n, x_{n+1})\right\}. \end{aligned}$$

If

$$\max\{p(x_{n-1},x_n),p(x_n,x_{n+1})\} = p(x_n,x_{n+1}), \tag{3.5}$$

for some $n \in \mathbb{N}$ then from (3.2) and (3.3), we have
$p(x_n,x_{n+1})) \leq M(x_{n-1},x_n) = p(x_n,x_{n+1})$, which is possible only if $\beta(p(x_n,x_{n+1})) = 1 \Rightarrow p(x_n,x_{n+1}) = 0$ a contradiction.
Therefore, we have $M(x_{n-1},x_n) = \max\{p(x_{n-1},x_n),p(x_n,x_{n+1})\} = p(x_n,x_{n+1})$ is a contradiction.
Thus, we have $M(x_{n-1},x_n) = \max\{p(x_{n-1},x_n),p(x_n,x_{n+1})\} = p(x_{n-1},x_n)$ for all $n \in \mathbb{N}$ and hence,

$$p(x_n,x_{n+1}) < p(x_{n-1},x_n), \tag{3.6}$$

for all $n \in \mathbb{N}$.
Thus it follows that $\{p(x_n,x_{n+1})\}$ is a non-negative, decreasing sequence of real numbers. Suppose that $\lim_{n\to\infty} p(x_n,x_{n+1}) = r, r \geq 0$. Now we prove that $r = 0$. Assume that $r > 0$. Now by (3.2), when n is even

$$\begin{aligned} p(x_n,x_{n+1}) &\leq sp(x_n,x_{n+1}) \\ &\leq \alpha(x_{n-1},Tx_{n-1})\alpha(x_n,Sx_n)sp(Tx_{n-1},Sx_n)) \\ &\leq \beta(p(x_{n-1},x_n))(p(x_{n-1},x_n)) \\ &\leq p(x_{n-1},x_n), \end{aligned}$$

for all even n.
When n is odd

$$\begin{aligned} p(x_n,x_{n+1}) &\leq sp(x_n,x_{n+1}) \\ &\leq \alpha(x_{n-1},Sx_{n-1})\alpha(x_n,Tx_n)sp(Sx_{n-1},Tx_n)) \\ &\leq \beta(p(x_{n-1},x_n))(p(x_{n-1},x_n)) \\ &\leq p(x_{n-1},x_n), \end{aligned}$$

for all odd n.

$$\begin{aligned} \therefore p(x_n,x_{n+1}) &\leq \beta(p(x_{n-1},x_n))(p(x_{n-1},x_n)) \\ &\leq p(x_{n-1},x_n), \ \ \forall\, n \in \mathbb{N}. \end{aligned}$$

On taking limits as $n \to \infty$, we have,

$$\begin{aligned} \lim_{n\to\infty} p(x_n,x_{n+1}) &\leq \lim_{n\to\infty} \beta(p(x_{n-1},x_n))(p(x_{n-1},x_n)) \\ &\leq \lim_{n\to\infty} p(x_n,x_{n+1}) \\ &\Rightarrow r \leq \lim_{n\to\infty} \beta(p(x_{n-1},x_n))r \leq r \\ &\Rightarrow \lim_{n\to\infty} \beta(p(x_{n-1},x_n)) = 1 \\ &\Rightarrow \lim_{n\to\infty} (p(x_{n-1},x_n)) = 0 \\ &\Rightarrow r = 0, \end{aligned}$$

a contradiction our assumption $r > 0$. Hence $r = 0$.

$$\therefore r = \lim_{n\to\infty} p(x_n,x_{n+1}) = 0. \tag{3.7}$$

Now we claim sequence $\{x_n\}$ is a Cauchy sequence. Assume that $\{x_n\}$ is not a Cauchy sequence. Then by Lemma 3.5, $\exists\ \epsilon > 0$ and sequences $\{x_{n_k}\}$, $\{x_{m_k}\}$; $m_k > n_k > k$ such that $p(x_{m_k}, x_{n_k}) \geq \epsilon$ and $p(x_{m_k-1}, x_{n_k}) < \epsilon$.
Let us observe the following cases:
Case(i): Let m_k is even and n_k is odd

$$\begin{aligned}
\therefore s\epsilon &\leq sp(x_{m_k}, x_{n_k}) \\
&= sp(Tx_{m_k-1}, Sx_{n_k-1})\} \\
&\leq \alpha(x_{m_k-1}, Tx_{m_k-1})\alpha(x_{n_k-1}, Sx_{n_k-1})\beta(M(x_{m_k-1}, x_{n_k-1}))M(x_{m_k-1}, x_{n_k-1}) \\
&\leq \beta(M(x_{m_k-1}, x_{n_k-1})M(x_{m_k-1}, x_{n_k-1}) < M(x_{m_k-1}, x_{n_k-1}),
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
M(x_{m_k-1}, x_{n_k-1}) &= \max[p(x_{m_k-1}, x_{n_k-1}), p(x_{n_k-1}, Sx_{n_k-1}), p(x_{m_k-1}, Tx_{m_k-1}), \\
&\frac{1}{2s}[\{p(x_{m_k-1}, Sx_{n_k-1}) + p(Tx_{m_k-1}, x_{n_k-1})\}] \\
&= \max[p(x_{m_k-1}, x_{n_k-1}), p(x_{n_k-1}, x_{n_k}), p(x_{m_k-1}, x_{m_k}), \\
&\frac{1}{2s}[\{p(x_{m_k-1}, x_{n_k}) + p(x_{m_k}, x_{n_k-1})\}] \\
&\leq \max[p(x_{m_k-1}, x_{n_k-1}), p(x_{n_k-1}, x_{n_k}), p(x_{m_k-1}, x_{m_k}), \\
&\frac{1}{2s}[\{sp(x_{m_k-1}, x_{n_k-1}) + sp(x_{n_k-1}, x_{n_k}) - p(x_{n_k-1}, x_{n_k-1}) \\
&+ sp(x_{m_k-1}, x_{n_k-1}) + sp(x_{m_k-1}, x_{m_k}) - p(x_{m_k-1}, x_{m_k-1})\}] \\
&\leq \max[p(x_{m_k-1}, x_{n_k-1}), p(x_{n_k-1}, x_{n_k}), p(x_{m_k-1}, x_{m_k}), \\
&\frac{1}{2s}[\{2sp(x_{m_k-1}, x_{n_k-1}) + sp(x_{n_k-1}, x_{n_k}) + sp(x_{m_k}, x_{m_k-1})\}] \\
&= p(x_{m_k-1}, x_{n_k-1}) + \frac{1}{2}p(x_{n_k-1}, x_{n_k}) + \frac{1}{2}p(x_{m_k}, x_{m_k-1}) \\
&\leq sp(x_{m_k-1}, x_{n_k}) + sp(x_{n_k}, x_{n_k-1}) - p(x_{n_k}, x_{n_k}) + \frac{1}{2}p(x_{n_k-1}, x_{n_k}) + \frac{1}{2}p(x_{m_k}, x_{m_k-1}) \\
&\leq sp(x_{m_k-1}, x_{n_k}) + sp(x_{n_k}, x_{n_k-1}) + \frac{1}{2}p(x_{n_k-1}, x_{n_k}) + \frac{1}{2}p(x_{m_k}, x_{m_k-1}) \\
&\leq s\epsilon + s\eta + \frac{1}{2}\eta + \frac{1}{2}\eta,
\end{aligned}$$

where
$p(x_{n_k-1}, x_{n_k}) < \eta$ and $p(x_{m_k}, x_{m_k-1}) < \eta$; $\eta \to 0$ as $k \to \infty$

$$\therefore s\epsilon \leq \beta(M(x_{m_k-1}, x_{n_k-1})(s\epsilon + s\eta + \eta). \tag{3.9}$$

Allowing $k \to \infty$,

$$\begin{aligned}
s\epsilon &\leq \lim_{k\to\infty} \beta(M(x_{m_k-1}, x_{n_k-1}) \lim_{k\to\infty} (s\epsilon + s\eta + \eta) \\
s\epsilon &\leq \lim_{k\to\infty} \beta(M(x_{m_k-1}, x_{n_k-1})(s\epsilon) \\
\therefore &\lim_{k\to\infty} \beta(M(x_{m_k-1}, x_{n_k-1})) = 1
\end{aligned}$$

$$\therefore \lim_{k\to\infty} M(x_{m_k-1}, x_{n_k-1}) = 0,$$

then by (3.9) $s\epsilon \le 0$, a contradiction.
Case(ii): Let m_k is odd and n_k is odd

$$\begin{aligned}\therefore sp(x_{m_k}, x_{n_k+1}) &\le \alpha(x_{m_k-1}, Sx_{m_k-1})\alpha(x_{n_k}, Tx_{n_k})sp(Sx_{m_k-1}, Tx_{n_k}))\\ &\le \beta(M(x_{m_k-1}, x_{n_k})M(x_{m_k-1}, x_{n_k})\\ &< M(x_{m_k-1}, x_{n_k}),\end{aligned} \tag{3.10}$$

where

$$\begin{aligned} &M(x_{m_k-1}, x_{n_k})\\ &= \max\left[p(x_{m_k-1}, x_{n_k}), p(x_{m_k-1}, Sx_{m_k-1}), p(x_{n_k}, Tx_{n_k}), \frac{1}{2s}[\{p(Sx_{m_k-1}, x_{n_k}) \; + p(x_{m_k-1}, Tx_{n_k})\}\right]\\ &= \max\left[p(x_{m_k-1}, x_{n_k}), p(x_{m_k-1}, x_{m_k}), p(x_{n_k}, x_{n_k+1}), \frac{1}{2s}[\{p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k+1})\}\right]\\ &= p(x_{m_k-1}, x_{n_k}) \quad \text{or} \quad \frac{1}{2s}\left[\{p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k+1})\}\right]\end{aligned}$$

Suppose $M(x_{m_k-1}, x_{n_k}) = p(x_{m_k-1}, x_{n_k}) < \epsilon$.
But

$$\begin{aligned}\epsilon &\le p(x_{m_k}, x_{n_k}) \le sp(x_{m_k}, x_{n_k+1}) + sp(x_{n_k+1}, x_{n_k}) - p(x_{n_k+1}, x_{n_k+1})\\ &\le sp(x_{m_k}, x_{n_k+1}) + s\eta \text{ where } \eta > 0 \ni p(x_{n_k+1}, x_{n_k}) < \eta \qquad (3.11)\\ &\Rightarrow \epsilon - s\eta \le sp(x_{m_k}, x_{n_k+1}), \qquad (3.12)\end{aligned}$$

$$\begin{aligned}\therefore \epsilon - s\eta \le sp(x_{m_k}, x_{n_k+1}) &\le \alpha(x_{m_k-1}, Sx_{m_k-1})\alpha(x_{n_k}, Tx_{n_k})sp(Sx_{m_k-1}, Tx_{n_k}))\\ &\le \beta(p(x_{m_k-1}, x_{n_k})p(x_{m_k-1}, x_{n_k})\\ &< p(x_{m_k-1}, x_{n_k}) < \epsilon\end{aligned} \tag{3.13}$$

Allowing $k \to \infty$, then $\eta \to 0$

$$\begin{aligned}&\therefore \epsilon \le \lim_{k\to\infty} \beta(p(x_{m_k-1}, x_{n_k}))(\epsilon) \le \epsilon \text{ and } \lim_{k\to\infty} p(x_{m_k-1}, x_{n_k}) = \epsilon\\ &\therefore \lim_{k\to\infty} \beta(p(x_{m_k-1}, x_{n_k})) = 1\\ &\Rightarrow \lim_{k\to\infty} p(x_{m_k-1}, x_{n_k}) = 0\\ &\Rightarrow \epsilon = 0,\end{aligned}$$

a contradiction.
Suppose $M(x_{m_k-1}, x_{n_k}) = \frac{1}{2s}\ [\{p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k+1})\}]$.
On the other hand

$$\begin{aligned} p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k+1}) &\leq sp(x_{m_k}, x_{n_k+1}) + sp(x_{n_k+1}, x_{n_k}) - \mathrm{p}(\mathrm{x}_{n_k+1}, x_{n_k+1}) + sp(x_{m_k-1}, x_{m_k}) \\ &\quad + sp(x_{m_k}, x_{n_k+1}) - p(x_{m_k}, x_{m_k}) \\ &\leq sp(x_{m_k}, x_{n_k+1}) + sp(x_{n_k+1}, x_{n_k}) + sp(x_{m_k}, x_{n_k+1}) + sp(x_{m_k-1}, x_{m_k}) \\ &\leq 2sp(x_{m_k}, x_{n_k+1}) + 2s\eta \leq 2s\epsilon + 2s\eta, \end{aligned}$$

where $p(x_{m_k-1}, x_{m_k}) \leq \eta$ and $p(x_{n_k}, x_{n_k+1}) \leq \eta$ for some $\eta > 0$ for large k,

$$\therefore \frac{1}{2s}[\{p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k+1})\}] \leq \epsilon + \eta. \tag{3.14}$$

Therefore,

$$M(x_{m_k-1}, x_{n_k}) = \frac{1}{2s}[\{p(x_{m_k}, x_{n_k}) + p(x_{m_k-1}, x_{n_k+1})\}] \leq \epsilon + \eta.$$

$\therefore$ From (3.12), (3.13) and (3.14),

$$\begin{aligned} \epsilon - s\eta &\leq sp(x_{m_k}, x_{n_k+1}) \\ &\leq \beta(M(x_{m_k-1}, x_{n_k}))(M(x_{m_k-1}, x_{n_k})) \\ &\leq M(x_{m_k-1}, x_{n_k}) \\ &\leq \epsilon + \eta. \end{aligned}$$

Allowing $k \to \infty$, then $\eta \to 0$

$$\begin{aligned} &\therefore \epsilon \leq \lim_{k\to\infty} \beta(M(x_{m_k-1}, x_{n_k})) \lim_{k\to\infty} M(x_{m_k-1}, x_{n_k}) \leq \epsilon \text{ and } \lim_{k\to\infty} M(x_{m_k-1}, x_{n_k}) = \epsilon \\ &\therefore \lim_{k\to\infty} \beta(M(x_{m_k-1}, x_{n_k})) = 1 \\ &\Rightarrow \lim_{k\to\infty} M(x_{m_k-1}, x_{n_k}) = 0 \\ &\Rightarrow \epsilon = 0, \end{aligned}$$

a contradiction.
Similarly the other two cases can be discussed.
$\therefore \{x_n\}$ is a Cauchy sequence. Hence $\lim\limits_{n,m\to\infty} p(x_n, x_m)$ exists and is equal to 0 (by (3.7) and Lemma 3.5).
Since (X, p) is complete, $\therefore \{x_n\} \to y$ for some $y \in X$, then

$$0 = \lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, y) = p(y, y).$$

Let n be even and

$$\alpha(y, Ty) \geq 1 \quad \text{(by } (iii)). \tag{3.15}$$

Now,

$$\begin{aligned} sp(Sx_n, Ty) &\leq \alpha(x_n, Sx_n)\alpha(y, Ty)sp(Sx_n, Ty) \\ &\leq \beta(M(x_n, y))M(x_n, y) < M(x_n, y) \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} M(x_n, y) &= \max\left\{p(x_n, y), p(y, Ty), p(x_n, Sx_n), \frac{1}{2s}[p(x_n, Ty) + p(Sx_n, y)]\right\} \\ &= \max\left\{p(x_n, y), p(y, Ty), p(x_n, x_{n+1}), \frac{1}{2s}[p(x_n, Ty) + p(x_{n+1}, y)]\right\} \\ &\leq \max\left\{p(x_n, y), p(y, Ty), p(x_n, x_{n+1}), \frac{1}{2s}[sp(x_n, y) + sp(y, Ty) - p(y, y) + p(x_{n+1}, y)]\right\} \\ &= p(y, Ty) \quad \text{for large} \quad n. \qquad (3.17) \\ &\therefore sp(Sx_n, Ty) = sp(x_{n+1}, Ty) < M(x_n, y) = p(y, Ty). \end{aligned}$$

But

$$\lim_{n\to\infty} x_{n+1} = y, \tag{3.18}$$

$\therefore$ By Lemma 3.5,

$$\lim_{n\to\infty} sp(Sx_n, Ty) = \lim_{n\to\infty} sp(x_{n+1}, Ty) = p(y, Ty). \tag{3.19}$$

Now by (3.12)

$$sp(Sx_n, Ty) \leq \beta(M(x_n, y))M(x_n, y) < M(x_n, y).$$

Allowing $n \to \infty$,

$$\begin{aligned} &\lim_{n\to\infty} sp(Sx_n, Ty) \leq \lim_{n\to\infty} \beta(M(x_n, y))M(x_n, y) \leq \lim_{n\to\infty} M(x_n, y) \\ &\Rightarrow p(y, Ty) \leq \lim_{n\to\infty} \beta(M(x_n, y))p(y, Ty) \leq p(y, Ty). \end{aligned}$$

Therefore

$$\begin{aligned} &\lim_{n\to\infty} \beta(M(x_n, y)) = 1 \\ \Rightarrow &\lim_{n\to\infty} M(x_n, y) = 0 \\ \Rightarrow &p(y, Ty) = 0 \Rightarrow y = Ty. \end{aligned}$$

Therefore y is a fixed point of T.
Let n be odd and

$$\alpha(y, Sy) \geq 1 \quad (\text{by } (iii)). \tag{3.20}$$

Now,

$$\begin{aligned} sp(Tx_n, Sy) &\le \alpha(x_n, Tx_n)\alpha(y, Sy)sp(Tx_n, Sy) \\ &\le \beta(M(x_n, y))M(x_n, y) < M(x_n, y), \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} M(x_n, y) &= \max\left\{p(x_n, y), p(y, Sy), p(x_n, Tx_n), \frac{1}{2s}[p(x_n, Sy) + p(Tx_n, y)]\right\} \\ &= \max\left\{p(x_n, y), p(y, Sy), p(x_n, x_{n+1}), \frac{1}{2s}[p(x_n, Sy) + p(x_{n+1}, y)]\right\} \\ &\le \max\left\{p(x_n, y), p(y, Sy), p(x_n, x_{n+1}), \frac{1}{2s}[sp(x_n, y) + sp(y, Sy) - p(y, y) + p(x_{n+1}, y)]\right\}, \\ &= p(y, Sy) \quad \text{for large} \quad n. \qquad (3.22) \\ &\therefore sp(Tx_n, Sy) = sp(x_{n+1}, Sy) < M(x_n, y) = p(y, Sy). \end{aligned}$$

But

$$\lim_{n\to\infty} x_{n+1} = y. \tag{3.23}$$

$\therefore$ By Lemma 3.5,

$$\lim_{n\to\infty} sp(Tx_n, Sy) = \lim_{n\to\infty} sp(x_{n+1}, Sy) = p(y, Sy). \tag{3.24}$$

Now by (3.12)

$$sp(Tx_n, Sy) \le \beta(M(x_n, y))M(x_n, y) < M(x_n, y).$$

Allowing $n \to \infty$,

$$\begin{aligned} &\lim_{n\to\infty} sp(Tx_n, Sy) \le \lim_{n\to\infty} \beta(M(x_n, y))M(x_n, y) \le \lim_{n\to\infty} M(x_n, y) \\ &\Rightarrow p(y, Sy) \le \lim_{n\to\infty} \beta(M(x_n, y))p(y, Sy) \le p(y, Sy). \end{aligned}$$

Therefore

$$\begin{aligned} &\lim_{n\to\infty} \beta(M(x_n, y)) = 1 \\ &\Rightarrow \lim_{n\to\infty} M(x_n, y) = 0 \\ &\Rightarrow p(y, Sy) = 0 \Rightarrow y = Sy. \end{aligned}$$

Therefore y is a fixed point of S. Hence S, T has a unique common fixed point. □

Now we state and prove our second main result.

Theorem 3.8. *Let $(X, \leq, p)$ be a complete partially ordered partial b-metric space with $s \geq 1$ and let $\alpha : X \times X \to [0, \infty)$ be a function such that $\alpha(x, x) \geq 1 \ \forall\ x \in X$. Let S, T be a pair of weakly increasing self maps on X. Suppose that there exists $\beta \in \Omega$ such that $\alpha(x, y)p(Sx, Ty) \leq \beta(M(x, y))M(x, y)$ for all $x, y \in X$, where*

$$M(x,y) = \max\left\{p(x,y), p(x,Sx), p(y,Ty), \frac{1}{2s}[p(x,Ty) + p(Sx,y)]\right\}.$$

Assume that
(i) S, T are $\alpha-$admissible,
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$,
(iii) X is α regular and for every sequence $\{x_n\} \subset X$ such that $\alpha(x_n, x_{n+1}) \geq 1 \ \ \forall\ n \in \mathbb{N} \cup \{0\}$, we have $\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$,
then S, T have a unique fixed point in X.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Define the sequence $\{x_n\}$ in X by $x_{2n} = Tx_{2n-1}$ and $x_{2n-1} = Sx_{2n-2} \ \ \forall\ n \in \mathbb{N}$. We have by Theorem 3.7, $\{x_n\}$ is a Cauchy sequence such that $\lim\limits_{n\to\infty} p(x_n, x_{n+1}) = 0$.
$\therefore \lim\limits_{n,m\to\infty} p(x_n, x_m)$ exists and equal to 0. Since $(X, \leq, p)$ is complete.
$\therefore \{x_n\} \to z$ for some $z \in X$ such that

$$0 = \lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, z) = p(z, z). \tag{3.25}$$

Since X is regular, therefore there exists a sub sequences $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\alpha(x_{n_k}, z) \geq 1 \ \forall\ k \in \mathbb{N}. \tag{3.26}$$

Let n_k be even

$$\begin{aligned}\therefore sp(x_{n_k+1}, Tz) &\leq \alpha(x_{n_k}, z)sp(Sx_{n_k}, Tz)\\ &\leq \beta(M(x_{n_k}, z))M(x_{n_k}, z) < M(x_{n_k}, z),\end{aligned} \tag{3.27}$$

where

$$\begin{aligned}M(x_{n_k}, z) &= \max\left\{p(x_{n_k}, z), p(x_{n_k}, Sx_{n_k}), p(z, Tz), \frac{1}{2s}[p(x_{n_k}, Tz) + p(Sx_{n_k}, z)]\right\}\\ &= \max\left\{p(x_{n_k}, z), p(x_{n_k}, x_{n_k+1}), p(z, Tz), \frac{1}{2s}[p(x_{n_k}, Tz) + p(x_{n_k+1}, z)]\right\}\\ &\leq \max\left\{p(x_{n_k}, z), p(x_{n_k}, x_{n_k+1}), p(z, Tz), \frac{1}{2s}[sp(x_{n_k}, z) + sp(z, Tz) - p(z, z) + p(x_{n_k+1}, z)]\right\}\\ &\leq \max\left\{p(x_{n_k}, z), p(x_{n_k}, x_{n_k+1}), p(z, Tz), \frac{1}{2s}[sp(x_{n_k}, z) + sp(z, Tz) + p(x_{n_k+1}, z)]\right\}\\ &= p(z, Tz) \quad \text{for large } k. \qquad (3.28)\\ &\Rightarrow sp(x_{n_k+1}, Tz) \leq p(z, Tz) \text{ and } \{x_n\} \to z. \qquad (3.29)\end{aligned}$$

$\therefore$ By Lemma 3.6,

$$\begin{aligned}
&\lim_{n\to\infty} sp(x_n, z) = p(z, Tz) \\
&\therefore p(z, Tz) \leq \beta(p(z, Tz))p(z, Tz) < p(z, Tz) \\
&\Rightarrow p(z, Tz) = 0 \qquad (3.30)\\
&\therefore z = Tz,
\end{aligned}$$

$\therefore$ z is a fixed point of T in X.
Let n_k be odd

$$\begin{aligned}
\therefore sp(x_{n_k+1}, Sz) &\leq \alpha(x_{n_k}, z)sp(Tx_{n_k}, Sz) \\
&\leq \beta(M(x_{n_k}, z))M(x_{n_k}, z) < M(x_{n_k}, z), \qquad (3.31)
\end{aligned}$$

where

$$\begin{aligned}
M(x_{n_k}, z) &= \max\left\{p(x_{n_k}, z), p(x_{n_k}, Tx_{n_k}), p(z, Sz), \frac{1}{2s}[p(x_{n_k}, Sz) + p(Tx_{n_k}, z)]\right\} \\
&= \max\left\{p(x_{n_k}, z), p(x_{n_k}, x_{n_k+1}), p(z, Sz), \frac{1}{2s}[p(x_{n_k}, Sz) + p(x_{n_k+1}, z)]\right\} \\
&\leq \max\left\{p(x_{n_k}, z), p(x_{n_k}, x_{n_k+1}), p(z, Sz), \frac{1}{2s}[sp(x_{n_k}, z) + sp(z, Sz) - p(z, z) + p(x_{n_k+1}, z)]\right\} \\
&\leq \max\left\{p(x_{n_k}, z), p(x_{n_k}, x_{n_k+1}), p(z, Sz), \frac{1}{2s}[sp(x_{n_k}, z) + sp(z, Sz) + p(x_{n_k+1}, z)]\right\} \\
&= p(z, Sz) \quad \text{for large } k \qquad (3.32)\\
&\Rightarrow sp(x_{n_k+1}, Sz) \leq p(z, Sz) \text{ and } \{x_n\} \to z. \qquad (3.33)
\end{aligned}$$

$\therefore$ By Lemma 3.6,

$$\begin{aligned}
&\lim_{n\to\infty} sp(x_n, z) = p(z, Sz) \\
&\therefore p(z, Sz) \leq \beta(p(z, Sz))p(z, Sz) < p(z, Sz) \\
&\Rightarrow p(z, Sz) = 0 \qquad (3.34)\\
&\therefore z = Sz.
\end{aligned}$$

$\therefore$ z is a fixed point of S in X.
Assume that u and v, with $u \neq v$ are two fixed points of S, T. Then $Su = Tu = u$ and $Sv = Tv = v$,

$$0 < p(u, v) \leq sp(u, v) \leq \alpha(u, v)sp(Tu, Sv) \leq \beta(M(u, v))M(u, v) < M(u, v),$$

where

$$\begin{aligned}
M(u, v) &= \max\left\{p(u, v), p(u, Tu), p(v, Sv), \frac{1}{2s}[p(u, Sv) + p(Tu, v)]\right\} \\
&= p(u, v), \qquad (3.35)
\end{aligned}$$

$0 < p(u, v) \leq \beta(M(u, v))M(u, v) < M(u, v) = p(u, v)$, which is a contradiction. Therefore, we get $p(u, v) = 0 \Rightarrow u = v$. Hence S, T have a unique fixed point in X. □

Now we state and prove our third main result.

Theorem 3.9. *Let $(X, \leq, p)$ be a complete partially ordered partial b-metric space with $s \geq 1$ and let $S, T : X \to X$ be a pair self maps weakly increasing. Let $\alpha : X \times X \to [0, \infty)$ be a function. Suppose there exists $\beta \in \Omega$ such that $sp(Sx, Ty) \leq \beta(M(x, y))M(x, y)$ for all $x, y \in X$ with $x \leq y$, where*

$$M(x, y) = \max\left\{p(x, y), p(x, Sx), p(y, Ty), \frac{1}{2s}[p(x, Ty) + p(Sx, y)]\right\}.$$

Assume that
(i) there exists $x_0 \in X$ such that $x_0 \leq Sx_0$,
(ii) X is such that, if a non-decreasing sequence $\{x_n\}$ converges x, then there exists a sub sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \leq x \ \forall\ k \in \mathbb{N}$,
(iii) x, y are comparable whenever $x, y \in Fix\{S, T\}$,
then S, T have a unique fixed point x in X.

Proof. Define mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if} \quad x \leq y \\ 0, & \text{otherwise.} \end{cases}$$

Since S and T are weakly increasing, $x_1 = Sx_0 \leq TSx_0 = Tx_1 = x_2 \leq STx_1 = Sx_2 = x_3 \cdots$.
$\therefore x_1 \leq x_2 \leq x_3 \leq \cdots$. Thus $\{x_n\}$ is non-decreasing.
We have by (i) there exists $x_0 \in X$ be such that $x_0 \leq Sx_0 \Rightarrow \alpha(x_0, Sx_0) \geq 1$ which is the condition (ii) of Theorem 3.8.
Without loss of generality, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By using the α-admissibility of T, we have $\alpha(x_0, x_1) = \alpha(x_0, Sx_0) \geq 1 \Rightarrow \alpha(x_1, x_2) = \alpha(Sx_0, Tx_1) \geq 1$. Now, by mathematical induction, it is easy to see that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$.
$\therefore$ S, T are $\alpha-$admissible, which is the condition (i) of Theorem 3.8.
Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1 \quad \forall\ n \in \mathbb{N} \cup \{0\}$, and $x_n \to x \in X$ as $n \to \infty$. By definition of α, we have $x_n \leq x_{n+1} \ \forall\ n \in \mathbb{N} \cup \{0\}$.
$\therefore \{x_n\}$ is non-decreasing.
$\therefore$ By (ii) of this theorem, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \leq x \ \forall\ k \in \mathbb{N}$ and hence X is α-regular. Further, $\alpha(x_m, x_n) \geq 1 \quad \forall\ m, n \in \mathbb{N}$ with $m < n$. Hence (iii) of Theorem 3.8 holds.
By condition (iii) of this theorem, $x, y \in Fix\{S, T\} \Rightarrow x \leq y \Rightarrow \alpha(x, y) \geq 1$.
Thus hypothesis of Theorem 3.8 holds. Hence by Theorem 3.8, S, T have a unique common fixed point in X. □

Corollary 3.10. *Let $(X, \leq, p)$ be a complete partially ordered partial b-metric space with $s \geq 1$ and let $S, T : X \to X$ be a pair of weakly increasing self maps. Let $\alpha : X \times X \to [0, \infty)$ be a function such that $\alpha(x, y) = 1 \ \forall\ x, y \in X$. Suppose there exists $\beta \in \Omega$ such that $sp(Sx, Ty) \leq \beta(M(x, y))M(x, y)$ for all $x, y \in X$, where*

$$M(x, y) = \max\left\{p(x, y), p(x, Sx), p(y, Ty), \frac{1}{2s}[p(x, Ty) + p(Sx, y)]\right\}.$$

Then S, T have a unique common fixed point z in X.

Now we give an example in support of Corollary 3.10.

Example 3.11. Let $X = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{10}\right\}$ with usual ordering.
Define

$$p(x,y)=\begin{cases}0, \text{ if } x=y\\ 1, \text{ if } x\neq y\in\{0,1\}\\ |x-y|, \text{ if } x,y\in\left\{0,\frac{1}{2},\frac{1}{4},\frac{1}{6},\frac{1}{8},\frac{1}{10}\right\}\\ 4, \text{ otherwise.}\end{cases}$$

Clearly, $(X,\leq,p)$ is a partially ordered partial b-metric space with coefficient $s=\frac{8}{3}$. (P. Kumam *et al.* [14]) Define $T:X\to X$ by

$$T1=T\frac{1}{3}=T\frac{1}{5}=T\frac{1}{7}=T\frac{1}{9}=0\ ;\ T0=T\frac{1}{2}=T\frac{1}{4}=T\frac{1}{6}=T\frac{1}{8}=T\frac{1}{10}=\frac{1}{4}\Rightarrow T(X)=\left\{0,\frac{1}{4}\right\}.$$

Define $S:X\to X$ by

$$S1=S\frac{1}{3}=S\frac{1}{5}=S\frac{1}{7}=S\frac{1}{9}=S0=S\frac{1}{2}=S\frac{1}{4}=S\frac{1}{6}=S\frac{1}{8}=S\frac{1}{10}=\frac{1}{4}\Rightarrow S(X)=\left\{\frac{1}{4}\right\},$$

and

$$\beta(t)=\begin{cases}\frac{1}{1+t}, & \text{if } \ t\in(0,\infty)\\ 0, & \text{if } t=0,\end{cases}$$

$\alpha(x,y)=1\ \forall\ x,y\in X$.
Let $A=\left\{0,\frac{1}{2},\frac{1}{4},\frac{1}{6},\frac{1}{8},\frac{1}{10}\right\}$ and $B=\left\{1,\frac{1}{3},\frac{1}{5},\frac{1}{7},\frac{1}{9}\right\}\Rightarrow T(A)=\frac{1}{4}$, $T(B)=0$ and $S(A)=\frac{1}{4}=S(B)$.
For $x,y\in X$ and $p(x,y)\neq 0\Rightarrow x\neq y$, then following are the cases
(i) For $x,y\in A\Rightarrow Sx=Ty=\frac{1}{4}\Rightarrow sp(Sx,Ty)=0$,

$$\therefore\ sp(Sx,Ty)\leq\beta(M(x,y))M(x,y) \text{ for all } x,y\in A;$$

(ii) For $x,y\in B\Rightarrow Sx=\frac{1}{4},Ty=0\Rightarrow sp(Sx,Ty)=(\frac{8}{3})(\frac{1}{4})=\frac{2}{3}$ where $M(x,y)=4\Rightarrow\beta(M(x,y))(M(x,y))=\frac{4}{5}$,

$$\therefore\ sp(Sx,Ty)\leq\beta(M(x,y))M(x,y) \text{ for all x,y } \in B;$$

(iii) For $x\in A,\ y\in B\Rightarrow Sx=\frac{1}{4},Ty=0\Rightarrow sp(Sx,Ty)=(\frac{8}{3})(\frac{1}{4})=\frac{2}{3}$ where $M(x,y)=4\Rightarrow\beta(M(x,y))M(x,y)=\frac{4}{5}$,

$$\therefore sp(Sx,Ty)\leq\beta(M(x,y))M(x,y);$$

(iv) For $x\in A,\ y\in B\Rightarrow Tx=Sy=\frac{1}{4}\Rightarrow sp(Tx,Sy)=0$,

$$\therefore sp(Sx,Ty)\leq\beta(M(x,y))M(x,y)$$
$$\therefore sp(Sx,Ty)\leq\beta(M(x,y))M(x,y) \text{ for all } x,y\in X.$$

Since $T\left(\frac{1}{4}\right)=S\left(\frac{1}{4}\right)=\frac{1}{4}$ and $\alpha\left(\frac{1}{4},T\frac{1}{4}\right)=1$. Therefore $\frac{1}{4}\in X$ is a fixed point. The hypothesis and conclusions Corollary 3.11 satisfied.

We observe that Theorems 2.13, 2.14 and 2.15 of V. La Rosa *et al.* [15] are true when $s = 1$ and $S = T = f$. Hence Theorems 2.13, 2.14 and 2.15 of V. La Rosa *et al.* [15] are corollaries of our main results.

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