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Some new fixed point results for multivalued maps

Latif Ur Rahman[a], Muhammad Arshad[a], Sami Ullah Khan[a,b,*], Ljiljana Gajic[c]

[a] *Department of Mathematics and Statistics, International Islamic university, H-10 Islamabad Pakistan.*

[b] *Department of Mathematics, Gomal University D. I. Khan, 29050, KPK, Pakistan.*

[c] *Department of Mathematics and informatics, 21000 Novi. Sad, trg D. Obradovica 4, Serbia.*

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Abstract

In this article, we apply the strong contractive mapping on the results of Rhoades [8] and establish some new fixed point results in spherically complete ultrametric space for multivalued maps and extend the corresponding results for the pair of Junck type mappings. The presented results unify, extend, and improve several results in the related literature.

Keywords: Ultra metric space, fixed point, spherically completeness, contractive mappings.
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1. Introduction

In 1922, Banach [1] established the first ever fundamental fixed point theorem which has played an important role in various fields of applied mathematical analysis. Due to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle. Rhoades [8], listed contractive mappings which were generalizations of Banach contraction principle. Roovij [9] introduced a new metric called ultrametric spaces and later on, Petalas *et al.* [5] and Gajic [2] proved various fixed point results in ultrametric space as a generalization of the Banach contraction principle. Over the couple of years, it has been generalized in various directions by famous mathematicians (see [1]–[9]).

Definition 1.1 ([9])**.** Let $(X, d^{'})$ be a metric space, if $d^{'}$ satifies strong triangular inequality i.e.

$$d^{'}(x, y) \leq \max\{d^{'}(x, z), d^{'}(z, y)\};$$

for all $x, y \in X$, then $d^{'}$ is called ultrametric on X and the pair $(X, d^{'})$ is called ultra metric space.

*Corresponding author

Email addresses: latifrahman1979@gmail.com (Latif Ur Rahman), marshadzia@iiu.edu.pk (Muhammad Arshad), gomal85@gmail.com (Sami Ullah Khan), gajic@dmi.uns.ac.rs (Ljiljana Gajic)

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Definition 1.2 ([8]). A self mapping $\check{T} : X \to X$ on the metric space $(X, d^{'})$ is said to be contractive mapping if

$$d^{'}(\check{T}x, \check{T}y) < \max\{d(y, \check{T}x), d^{'}(x, \check{T}y)\}; \text{ for all } x, y \in X,\ x \neq y.$$

Theorem 1.3 ([9]). *An ultrametric space is called spherically complete if the intersection of nested balls in X is non-empty.*

Gajic [2] proved the following result for multivalued mappings.

Theorem 1.4. *Suppose $(X, d^{'})$ be a spherically complete ultra metric space. If $\check{T} : X \to X$ is a mapping such that*

$$d^{'}(\check{T}x, \check{T}y) < \max\{d^{'}(x, y), d^{'}(x, \check{T}x), d^{'}(y, \check{T}y)\}; \textit{ for all } x, y \in X,\ x \neq y.$$

Then T has a unique fixed point in X.

Theorem 1.5 (Zorn's lemma). *Let S be a partially ordered set. If every totally ordered subset of S has an upper bound, then S contains a maximal element.*

Theorem 1.6. *An element $x \in X$ is said to be a coincidence point of $\check{S} : X \to X$ and $\check{T} : X \to 2^X_C$ if $\check{S}x \in \check{T}x$. We denote*

$$C(\check{S}, \check{T}) = \{\ x \in X \ \ /\ \check{S}x \in \check{T}x\ \};$$

the set of coincidence points of $\check{S}$ and $\check{T}$.

Theorem 1.7. *Suppose $(X, d^{'})$ be an ultrametric space, and $\hat{S} : X \to X$ and $\check{T} : X \to 2^X_C$, $\hat{S}$ and $\check{T}$ are said to be coincidentally commuting at $z \in X$ if $\hat{S}z \in \check{T}z$ implies $\hat{S}\check{T}z \subseteq \check{T}\hat{S}z$.*

Definition 1.8. Suppose $C(X)$ denote the class of all non empty compact subsets of X. for $A, B \in C(X)$ the Hausdorff metric is defined as

$$H(A, B) = \max\{\sup_{x \in B} d^{'}(x, A), \quad \sup_{y \in A} d^{'}(x, b)\};$$

where $d^{'}(x, A) = \inf\{d^{'}(x, a) : \ a \in A\ \}$.

Theorem 1.9 ([3]). *Suppose $(X, d^{'})$ be the spherically complete ultrametric space if $\check{T} : X \to 2^X_C$ is such that for any $x, y \in X$, $x \neq y$,*

$$H(\check{T}x, \check{T}y) < \max\{d^{'}(x, y), d^{'}(x, \check{T}x), d^{'}(y, \check{T}y)\};$$

Then $\check{T}$ has a fixed point. (i.e there exist $x \in X$, such that $x \in \check{T}x$).

2. The results

In this section, we apply strong contractive mapping on the results of Rhoades [8] and proved some new fixed point results in spherically complete ultrametric space for multivalued maps. Let us prove our first main result.

Theorem 2.1. *Suppose $(X, d^{'})$ be a complete ultrametric space if $\check{T} : X \to 2^X_C$ is such that for any $x, y \in X$, $x \neq y$, satisfying condition,*

$$H(\check{T}x, \check{T}y) < \max\{d^{'}(x, \check{T}y)\ , d^{'}(y, \check{T}x)\} \textit{ for all } x \neq y. \tag{2.1}$$

Then $\check{T}$ has a unique fixed point in X.

Proof. Let $S_a = (a, d^{'}(a, \check{T}a))$ is a closed sphere whose center is a and radius $d^{'}(a, \check{T}a) = \inf\limits_{d\in Ta} d^{'}(a, d^{'}) > 0$ for all $a \in X$. Let F is the collection of all such spheres on which the partial order is defined like $S_b \subseteq S_a$ iff $S_a \precsim S_b$. Let F_1 is totaly ordered subfamily of F, as $(X, d^{'})$ is spherically complete,

$$\bigcap_{S_a\in F_1} S_a = S \neq \phi.$$

Let $b \in S \implies b \in S_a$, as $S_a \in F_1$, hence $d^{'}(a, b) \leq d^{'}(a, Ta)$. Take $u \in Ta$ such that $d^{'}(a, u) = d^{'}(a, \check{T}a)$ (it is possible because $\check{T}a$ is non-empty compact set). If $a = b$, then $S_a = S_b$. Assume that $a \neq b$, and let $x \in S_b \implies$

$$\begin{aligned} d^{'}(x, b) \leq d^{'}(b, \check{T}b) \leq \inf_{v\in\check{T}b} d^{'}(b, v) &\leq \max\{d^{'}(b, a), d^{'}(a, u), \inf_{v\in\check{T}b} d^{'}(u, v)\} \\ &\leq \max\{d^{'}(a, \check{T}a), H(\check{T}a, \check{T}b)\}. \end{aligned}$$

Using (2.1), we get.

$$d^{'}(x, b) \leq \max\{d^{'}(a, \check{T}a), \max\{d^{'}(a, \check{T}b), d^{'}(b, \check{T}a)\}\}.$$

As $d^{'}(a, \check{T}b) \leq \max\{d^{'}(a, b), d^{'}(b, \check{T}b)\}$ and $d^{'}(b, \check{T}a) \leq \max\{d^{'}(b, a), d(a, \check{T}a)\}$. Therefore,

$$\begin{aligned} d^{'}(x, b) &\leq \max\{d^{'}(a, \check{T}a), \max\{d^{'}(a, b), d^{'}(b, \check{T}b)\}, \max\{d^{'}(b, a), d^{'}(a, \check{T}a)\}\}, \\ &= \max\{d^{'}(b, a), d^{'}(a, \check{T}a), d^{'}(b, \check{T}b)\} = d^{'}(a, \check{T}a), \\ d^{'}(x, b) &\leq d^{'}\check{T}(a, a). \end{aligned}$$

Now

$$\begin{aligned} &d^{'}(x, a) \leq \max\{d^{'}(x, b), d^{'}(b, a)\} \leq d^{'}(a, \check{T}a), \\ &d^{'}(x, a) \leq d^{'}(a, \check{T}a). \end{aligned}$$

So $x \in S_a$ and $S_b \subseteq S_a$ for all $S_a \in F_1$. Hence S_b is the upper bound of F for the family F_1 hence by the Zorn's lemma, F has a maximal element S_c for some $c \in X$. we are going to prove that $c \in Tc$. Suppose $c \notin \check{T}c$, then there exists $\overline{c} \in \check{T}c$ such that $d^{'}(c, \overline{c}) = d^{'}(c, \check{T}c)$.

$$\begin{aligned} d^{'}(\overline{c}, \check{T}\overline{c}) \leq H(\check{T}c, \check{T}\overline{c}) &< \max\{d^{'}(c, \check{T}\overline{c}), d^{'}(\overline{c}, \check{T}c\}, \\ &\leq \max\{\max\{d^{'}(c, \overline{c}), d^{'}(\overline{c}, \check{T}\overline{c})\}, \max\{d^{'}(\overline{c}, c\}, d^{'}(c, \check{T}c)\}, \\ &\leq \max\{d^{'}(c, \check{T}c), d^{'}(\overline{c}, \check{T}\overline{c}), \\ &= d^{'}(c, \check{T}c). \end{aligned}$$

This implies

$$d^{'}(\overline{c}, \check{T}\overline{c}) < d^{'}(c, \check{T}c).$$

Let $y \in S_{\overline{c}}$, implies that,

$$\begin{aligned} &d^{'}(y, \overline{c}) \leq d^{'}(\overline{c}, \check{T}\overline{c}) < d^{'}(c, \check{T}c), \\ &d^{'}(y, \overline{c}) < d^{'}(c, \check{T}c). \end{aligned}$$

As

$$d^{'}(y, c) \leq \max\{d^{'}(y, \overline{c}), d^{'}(\overline{c}, c)\} = d^{'}(c, \check{T}c),$$

$y \in S_c$ implies that $S_{\overline{c}} \subsetneqq S_c$, as $c \notin S_{\overline{c}}$ which is contradtion to the maximality of S_c, hence $c \in \check{T}c$. $\square$

Now, we extend the above result for Junck type multivalued functions.

Theorem 2.2. *Suppose* $(X, d^{'})$ *be a complete ultrametric space. Let* $\check{T} : X \to 2^X_C$ *and* $\hat{S}$ *is a self map on* X *which satisfies the contractive condition such that;*
(i) $\check{T}x \subseteq \hat{S}X$*, for all* $x, y \in X$;
(ii)

$$H(\check{T}x, \check{T}y) < \max\{d^{'}(\hat{S}x, \check{T}y), d^{'}(\hat{S}y, \check{T}x)\} \text{ for all } x \neq y; \tag{2.2}$$

(iii) $\hat{S}X$ *is spherically complete.*
Then there exists $z \in X$ *such that* $\hat{S}z \in \check{T}z$*. Further assume that*
(iv) $d^{'}(\hat{S}x, \hat{S}u) \leq H(\check{T}\hat{S}y; \check{T}u)$ *for all* $x, y, u \in X$ *with* $\hat{S}x \in \check{T}y$ *and* $\hat{S}$ *and* $\check{T}$ *are coincidentally commuting at c, then* $\hat{S}c$ *is the unique common fixed point of* $\hat{S}$ *and* $\check{T}$.

Proof. Let $B_a = (\hat{S}a, d^{'}(\hat{S}a, \check{T}a)) \bigcap \hat{S}X$ is a closed sphere whose center is $\hat{S}a$ and radius $d^{'}(\hat{S}a, \check{T}a) = \inf_{d^{'} \in \check{T}a} d^{'}(\hat{S}a, d^{'}) > 0$ for all $a \in X$ and let F be the collection of all such spheres on which the partial order is defined like $B_b \subseteq B_a$ iff $B_a \precsim B_b$. Let F_1 is totaly ordered subfamily of F, as $\hat{S}X$ is spherically complete

$$\bigcap_{B_a \in F_1} B_a = B \neq \phi.$$

Now $\hat{S}b \in B \implies \hat{S}b \in B_a$, as $B_a \in F_1$, hence $d^{'}(\hat{S}a, \hat{S}b) \leq d^{'}(a, \check{T}a)$. Take $u \in \check{T}a$ such that $d^{'}(\hat{S}a, u) = d^{'}(\hat{S}a, \check{T}a)$ (it is possible because $\check{T}a$ is non-empty compact set). If $\hat{S}a = \hat{S}b$ then $B_a = B_b$. Assume that $\hat{S}a \neq \hat{S}b$. For $x \in B_b \implies$

$$\begin{aligned} d^{'}(x, \hat{S}b) &\leq d^{'}(\hat{S}b, \check{T}b) \leq \inf_{v \in \check{T}b} d^{'}(\hat{S}b, v) \leq \max\{d^{'}(\hat{S}b, \hat{S}a), d^{'}(\hat{S}a, u), \inf_{v \in \check{T}b} d^{'}(u, v)\}, \\ &\leq \max\{d^{'}(\hat{S}b, \hat{S}a), d^{'}(\hat{S}a, \check{T}a), H(\check{T}a, \check{T}b) \\ &\leq \max\{d^{'}(\hat{S}a, \hat{S}b), d^{'}(\hat{S}a, \check{T}a), d^{'}(\hat{S}a, \check{T}b), d^{'}(\hat{S}b, \check{T}a)\}\}, \text{ using (2.2)}. \end{aligned}$$

As $d^{'}(\hat{S}a, \check{T}b) \leq \max\{d^{'}(\hat{S}a, \hat{S}b), d^{'}(\hat{S}b, \check{T}b)\}$ and $d^{'}(\hat{S}b, \check{T}a) \leq \max\{d^{'}(\hat{S}b, \hat{S}a), d^{'}(\hat{S}a, \check{T}a)\}$. Therefore,

$$\begin{aligned} d^{'}(x, \hat{S}b) &\leq \max\{d^{'}(\hat{S}b, \hat{S}a), d^{'}(\hat{S}a, \check{T}a), \max\{d^{'}(\hat{S}a, \hat{S}b), d^{'}(\hat{S}b, \check{T}b)\}, \max\{d^{'}(\hat{S}b, \hat{S}a), d^{'}(\hat{S}a, \check{T}a)\}\}, \\ &= d^{'}(\hat{S}a, \check{T}a). \end{aligned}$$

Now

$$d^{'}(x, \hat{S}a) \leq \max\{d^{'}(x, \hat{S}b), d^{'}(\hat{S}b, \hat{S}a)\} \leq d^{'}(\hat{S}a, \check{T}a),$$

implies

$$x \in B_a \text{ so } B_b \subseteq B_a \text{ for all } B_a \in F_1.$$

Hence B_b is the upper bound of F for the family F_1, hence by the Zorn's lemma F has a maximal element B_c for some $c \in X$. Now, we are going to prove that $\hat{S}c \in \check{T}c$. Suppose $\hat{S}c \notin \check{T}c$, then there exists $\hat{S}\overline{c} \in \check{T}c$ such that $d^{'}(\hat{S}c, \hat{S}\overline{c}) = d^{'}(\hat{S}c, \check{T}c) > 0$. So

$$\begin{aligned} d^{'}(\hat{S}\overline{c}, \check{T}\overline{c}) &\leq H(\check{T}c, \check{T}\overline{c}) < \max\{d^{'}(\hat{S}c, \check{T}\overline{c}), d^{'}(\hat{S}\overline{c}, \check{T}c)\}, \\ &\leq \max\{\max\{d^{'}(\hat{S}c, \hat{S}\overline{c}), d^{'}(\hat{S}\overline{c}, \check{T}\overline{c}), \max\{d^{'}(\hat{S}c, \hat{S}\overline{c}), d^{'}(\hat{S}c, \check{T}c)\}, \\ &= d^{'}(\hat{S}c, \check{T}c). \end{aligned}$$

This implies

$$d^{'}(\hat{S}\overline{c}, \check{T}\overline{c}) < d^{'}(\hat{S}c, \check{T}c).$$

Let $y \in B_{\overline{c}}$ then $d^{'}(y, \hat{S}\overline{c}) \leq d^{'}(\hat{S}\overline{c}, \check{T}\overline{c}) < d^{'}(\hat{S}c, \check{T}c)$. As

$$\begin{aligned} d^{'}(y,\ \hat{S}c) &\leq \max\{d^{'}(y, \hat{S}\overline{c}), d^{'}(\hat{S}\overline{c}, \hat{S}c)\}, \\ &= d^{'}(\hat{S}c, \check{T}c), \end{aligned}$$

where, $y \in B_c$ implies that $B_{\overline{c}} \subsetneqq B_c$, as $\hat{S}c \notin B_{\overline{c}}$, which is contradtion to the maximality of B_c, hence $\hat{S}c \in \check{T}c$. Further assume (iv) and write $\hat{S}c = e$. Then $e \in \check{T}c$.

$$d^{'}(e, \hat{S}e) = d^{'}(\hat{S}c, \hat{S}e) \leq H(\check{T}\hat{S}c, \check{T}e) = H(\check{T}e, \check{T}e) = 0.$$

This implies that $\hat{S}e = e$. From (iii), $e = \hat{S}e \in \hat{S}\check{T}c \subseteq \check{T}\hat{S}z = \check{T}e$. Thus $\hat{S}c = e$ is a common fixed point of $\hat{S}$ and $\check{T}$. Suppose $h \in X$, such that $e \neq h = \hat{S}h \in \check{T}h$. Using (iii)

$$\begin{aligned} d^{'}(e, h) = d^{'}(\hat{S}e, \hat{S}h) \leq H(\check{T}\hat{S}e, \check{T}h) &= H(\check{T}e, \check{T}h), \\ &< \max\{d^{'}(\hat{S}e, \check{T}h), d^{'}(\hat{S}h, \check{T}e)\}, \\ &= d^{'}(e, h). \end{aligned}$$

This implies that $e = h$. Thus $e = \hat{S}c$ is the unique common fixed point of $\hat{S}$ and $\check{T}$. □

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